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Regularization Auxiliary Problem Algorithm for Common Fixed Points of a Countably Infinite Family of Non-self Strictly Pseudocontractive Mappings

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Abstract

This paper deals with an iteration method combining the regularization process and the auxiliary problem principle for computing a common fixed point for a countably infinite family of non-self λ_i-strictly pseudocontractive mappings $\{T_i\}_{i=1}^{\infty}$ from a closed convex subset C into Hilbert space H.

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1. INTRODUCTION

Let H be a real Hilbert space with the scalar product and norm denoted by the symbols $\langle .,. \rangle$ and $\|.\|$, respectively, and let C be a closed convex subset in H. A mapping T of C into H is called λ-strictly pseudocontractive in the terminology of Browder and Petryshyn [4], if there exists a number $\lambda \in [0,1)$ such that

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + \lambda\|(I - T)(x) - (I - T)(y)\|^2,$$

where I is the identity operator in H. Clearly, when $\lambda = 0$, T is nonexpansive, i.e.,

$$\|T(x) - T(y)\| \leq \|x - y\|$$

for all $x, y \in D(T)$, the domain of definition of T. It means that the class of λ-strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings. Denote by $F(T)$ the set of fixed points of the mapping T in C, i.e., $F(T) = \{x \in C : x = T(x)\}$.

Let $\{T_i\}_{i=1}^{\infty}$ be a countally infinite family of λ_i-strictly pseudocontractive non-self mappings of C into H such that $\mathcal{F} := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. The problem studied in this paper is to find an element

$$u^* \in \mathcal{F}. \tag{1.1}$$

In the case that $T_i = I$ for $i > N = 1$, (1.1) is a problem of finding a fixed point for a λ-strictly pseudocontractive mapping in C and studied in [4], [22] and [17]. If $T_i = I$ for $i > N > 1$, then (1.1) is a problem of finding a common fixed point for a finite family of N λ_i-strictly pseudocontractive mappings T_i in C and studied in [27]. At this time, Buong and Son [5] used another approach, the regularization method, to solve (1.1) for the case that each T_i, maybe, is not a self mapping. Further, Buong [6] considered a regularization method to find the element u^* that is also a solution of the following variational inequality problem: find $u^* \in C$ such that

$$\langle A_0(u^*), v - u^* \rangle \geq 0 \quad \forall v \in C, \tag{1.2}$$

where A_0 is a Lipschitz continuous and monotone mapping from C into H. We denote by $V(C, A_0)$ the set of solutions for (1.2). Note that (1.2) can be regularized by the perturbative variational inequality problem: find $u_\alpha \in C$ such that

$$\langle A_0(u_\alpha) + \alpha u_\alpha, v - u_\alpha \rangle \geq 0 \quad \forall v \in C \tag{1.3}$$

(see [1]). Apart from the regularization algorithm (1.3), there are different algorithms to solve (1.2) a whole of which has been unified in the framework of the so-called auxiliary problem principle (see [8], [9]). This principle is constructed on the base of using an auxiliary function $\varphi : H \to (-\infty, \infty)$, which is chosen to be differentiable and strongly convex, and a sequence of positive numbers $\{\varepsilon_n\}_{n \geq 0}$. For some $x \in C$, we introduce the auxiliary problem

$$\min_{z \in C} \quad \varphi(z) + \langle \varepsilon_k \mathcal{A}_0(x) - \varphi'(x), z \rangle. \tag{1.4}$$

Let $z(x)$ denote the solution of this problem. It is characterized by the variational inequality

$$\langle \varphi'(z(x)) + \varepsilon_k A_0(x) - \varphi'(x), v - z(x) \rangle \geq 0 \quad \forall v \in C.$$

If $z(x)$ happens to be equal to x, then it is checked easily that $z(x)$ is also a solution of (1.2).

BASIC ALGORITHM
(i) At $k = 0$ start with z_0 and ε_0.
(ii) At step $k = n$, knowing z_n, compute $z_{n+1} = z(z_n)$ by solving the auxiliary problem (1.4) with x set to z_n

$$\min_{z\in C} \quad \varphi(z) + \langle \varepsilon_n A_0(z_n) - \varphi'(z_n), z\rangle. \tag{1.5}$$

(iii) Stop if $\|z_{n+1} - z_n\|$ is below some threshold. Otherwise, go back to the previous step.

Under other technical assumptuions and appropriate choice of the stepsize ε_n in (1.5), the basic algorithm converges when A_0 is strongly monotone [9] or has the Dunn property [26]. The basic algorithm converges also when the mapping A_0 is a gradient and monotone. In this case problem (1.5) is equivalent to minimization of convex functional (see [12]). However, it fails to convergence when the mapping A_0 is simply monotone, but not a gradient (see [13].)

The auxiliary problem principle is also used in [14], [15], [18], [20], [21], [23]-[25] to solve some other problems.

If $\lambda_i = 0$ for all $i \geq 1$, then $\{T_i\}_{i\geq 1}$ is a sequence of nonexpansive mappings from C into H. If, in addition, all T_i are the self mappings, i.e., $T_i : C \to C$, Maingé [16] considered the following iteration method

$$x_{n+1} = \alpha_n f(x_n) + \sum_{i\geq 1} w_{i,n} T_i x_n \quad n \geq 0,$$

where $f : C \to C$ is a given contraction with constant $\rho \in [0, 1)$, $x_0 \in C$ is a started point, $\{\alpha_n\} \subset (0, 1), w_{i,n} \geq 0$ for all $i \geq 1$ and $\sum_{i\geq 1} w_{i,n} = 1 - \alpha_n$ with some additional conditions.

Another result on a common fixed point for a countably infinite family of self nonexpansive mappings on C of a Hilbert space H is given in [7].

It is easy to see that if T_i is a λ_i-strictly pseudocontractive mapping, then $I - T_i$ is a $\tilde{\lambda}_i$-inverse strongly monotone mapping with $\tilde{\lambda}_i = (1 - \lambda_i)/2$, i.e.,

$$\langle (I - T_i)(x) - (I - T_i)(y), x - y\rangle \geq \tilde{\lambda}_i \|(I - T_i)(x) - (I - T_i)(y)\|^2 \quad \forall x, y \in C.$$

In this paper, on the base of Browder-Tikhonov regularization method in [1] and [2], and our results [5], [6] we present a new approximation method for solving (1.1).

We, firstly, construct a regularization solution u_α for (1.1) by solving the following variational inequality problem: find $u_\alpha \in C$ such that

$$\begin{gathered}\langle \sum_{i=1}^{\infty} \gamma_i A_i(u_\alpha) + \alpha u_\alpha, v - u_\alpha\rangle \geq 0 \quad \forall v \in C,\\ A_i = I - T_i, \quad i \geq 1,\end{gathered} \tag{1.6}$$

where $\alpha > 0$ is a small regularization parameter tending to zero, and $\{\gamma_i\}$ is a sequence of real numbers satisfying

$$\gamma_i > 0, \quad \sum_{i=1}^{\infty} \frac{\gamma_i}{\tilde{\lambda}_i} = \gamma < \infty. \tag{1.7}$$

Secondly, we consider a combination beetwen the method (1.6) and BASIC ALGORITHM to obtain a method, named regularization auxiliary problem algorithm. This combination was firstly realized by Baasansuren and Khan [2] for problem (1.2). Here, we consider the auxiliary problem in combination with the regularization method (1.6) for a countably infinite family of non-self λ_i-strictly pseudocontractive mappings T_i from C into H in the form:

We begin with an initial guess $z_0 \in C$ and initial parameters ε_0 and α_0, and solve the following problem

$$\min_{z\in C} \quad \varphi(z) + \langle \varepsilon_0(\mathcal{B}(z_0) + \alpha_0 z_0) - \varphi'(z_0), z\rangle, \quad \mathcal{B} = \sum_{i=1}^{\infty} \gamma_i A_i.$$

Let the functional φ be so chosen that the above minimization problem is uniquely solvable. We denote the (unique) solution by z_1 and continue further by replacing ε_0, α_0 and z_0 by ε_1, α_1 and z_1, respectively.

ALGORITHM A.

(i) At $k = 0$ start with z_0, ε_0 and α_0.

(ii) At step $k = n$ solve the following problem: find $z \in C$ such that

$$\min_{z\in C} \quad \varphi(z) + \langle \varepsilon_n(\mathcal{B}(z_n) + \alpha_n z_n) - \varphi'(z_n), z\rangle. \tag{1.8}$$

Let z_{n+1} be the solution.

(iii) Stop if $\|z_{n+1} - z_n\|$ is below some threshold. Otherwise, go back to the previous step.

For the sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$, we make the following assumption.

ASSUMPTION A. $0 < \varepsilon_n \leq 1; 0 < \alpha_{n+1} \leq \alpha_n \leq 1 : \alpha_n \to 0$ as $n \to \infty$.

$$\sum_{n=0}^{\infty} \varepsilon_n \alpha_n = \infty; \quad \sum_{n=0}^{\infty} \varepsilon_n^2 < \infty, \quad \sum_{n=1}^{\infty} \frac{(\alpha_n - \alpha_{n-1})^2}{\alpha_n^3 \varepsilon_n} < \infty.$$

The convergence of the algorithms (1.6) and (1.8) is proved in the next section.

2. MAIN RESULTS

We formulate the following facts needed in the proof of our results.

Let $G(u,v)$ be a bifunction from $C \times C \to (-\infty,+\infty)$. The equilibrium problem for G is to find $u^* \in C$ such that

$$G(u^*, v) \geq 0 \quad \forall v \in C. \tag{2.1}$$

Assume that the bifunction G satisfies the following standard properties.

Condition 2.1:

(A1) $G(u,u) = 0 \quad \forall u \in C$;
(A2) $G(u,v) + G(v,u) \leq 0 \quad \forall (u,v) \in C \times C$;
(A3) For every $u \in C$, $G(u,.) : C \to (-\infty,+\infty)$ is lower semicontinuous and convex;
(A4) $\overline{\lim}_{t\to+0} G((1-t)u + tz, v) \leq G(u,v) \quad \forall (u,z,v) \in C \times C \times C$;

Lemma 2.1 [11]. *Let C be a nonempty closed convex subset of H and G be a bifunction of $C \times C$ into $(-\infty,+\infty)$ satisfying Condition 2.1. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$G(z,v) + \frac{1}{r}\langle z - x, v - z\rangle \geq 0 \quad \forall v \in C.$$

Lemma 2.2 [11]. *Assume that $G : C \times C \to (-\infty,+\infty)$ satisfies Condition 2.1. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : G(z,v) + \frac{1}{r}\langle z - x, v - z\rangle \geq 0 \quad \forall v \in C\}.$$

Then, the following statements hold:
(i) T_r is single-valued;
(ii) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(iii) $F(T_r) = EP(G)$, the set of solutions for (2.1);
(iv) $EP(G)$ is closed and convex.

It is easy to see that T_r is a nonexpansive mapping.

Lemma 2.3 [19]. *Assume that T is a λ-strictly pseudocontractive mapping of a closed convex subset C of a Hilbert space H. Then $I - T$ is demiclosed at zero; that is whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)(x_n)\}$ strongly converges to zero, it follows $(I - T)(x) = 0$.*

Now, we are in the position to prove the following result.

Theorem 2.1. *Let C be a nonempty closed convex subset of real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of non-self λ_i-strictly pseudocontractive mappings from C into H such that $\mathcal{F} := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $\{\gamma_i\}$ satisfies condition (1.7). Then, we have:*
(i) For each $\alpha > 0$, problem (1.6) has a unique solution u_α.
(ii)

$$\lim_{\alpha \to 0} u_\alpha = u^*, u^* \in \mathcal{F}, \|u^*\| \leq \|y\| \quad y \in \mathcal{F}.$$

(iii)

$$\|u_\alpha - u_\beta\| \leq \frac{|\alpha - \beta|}{\alpha} \|u^*\|.$$

Proof.(i) Let y be a common fixed point of $\{T_i\}_{i \geq 1}$. Since

$$\gamma_i \|A_i(x)\| \leq \gamma_i \|A_i(x) - A_i(y)\| + \gamma_i \|A_i(y)\| \leq \gamma_i \frac{2}{1 - \lambda_i} \|x - y\|$$

and $\sum_{i=1}^{\infty} \frac{\gamma_i}{\lambda_i} = \gamma < +\infty$, the mapping $\mathcal{B}$ is well difined and $\sum_{i=1}^{\infty} \gamma_i A_i(x)$ converges absolutely for each $x \in C$. It is easy to see that $\mathcal{B}$ is Lipschitz continuous with the Lipschitz constant $L_{\mathcal{B}} = \gamma$, and monotone.

Set

$$G_i(u, v) = \langle \gamma_i A_i(u), v - u \rangle, \quad i \geq 1.$$

Then, problem (1.6) is equivalent to that: find $u_\alpha \in C$ such that

$$G_\alpha(u_\alpha, v) \geq 0 \quad \forall v \in C, \tag{2.2}$$

where

$$G_\alpha(u, v) = \tilde{G}(u, v) + \alpha \langle u, v - u \rangle$$
$$\tilde{G}(u, v) = \sum_{i=1}^{\infty} G_i(u, v) = \langle \mathcal{B}(u), v \rangle.$$

It is not difficult to verify that each $G_i(u, v), i \geq 1$, is a bifunction satisfying Condition 2.1. Therefore, $\tilde{G}(u, v)$ also is a bifunction satisfying Condition 2.1. By using Lemmas 2.1 and 2.2 with $(1/r) = \alpha > 0$ and $x = 0$, we can confirm that problem (2.2) (consequently (1.6)) has a unique solution u_α for each $\alpha > 0$.
(ii) Firstly, we shall prove that

$$\|u_\alpha\| \leq \|y\| \quad \forall y \in \mathcal{F}. \tag{2.3}$$

Since $y \in \mathcal{F}$, $A_i(y) = 0, i \geq 1$. Thus,

$$\langle \mathcal{B}(u_\alpha), y - u_\alpha \rangle + \alpha \langle u_\alpha, y - u_\alpha \rangle \geq 0 \quad \forall y \in \mathcal{F}. \tag{2.4}$$

As

$$\langle \gamma_i A_i(u_\alpha), y - u_\alpha \rangle \geq 0 \quad \forall y \in \mathcal{F}, i \geq 1,$$

from (2.4) it follows (2.3). Therefore, $\{u_\alpha\}$ is bounded. Then, there exists a subsequence $\{u_{\alpha_k}\}$ of the sequence $\{u_\alpha\}$ that converges weakly to some element $u^* \in C$. Now, we, firstly, prove $u^* \in \mathcal{F}$. From (2.4), the $(1-\lambda_l)/2$-inverse strongly monotone property of A_l and $A_i(y) = 0$ for all $i \geq 1$, it implies that

$$\begin{aligned} 0 \leq \gamma_l \frac{1-\lambda_l}{2}\|A_l(u_{\alpha_k})\|^2 &\leq \langle \gamma_l A_l(u_{\alpha_k}), u_{\alpha_k} - y\rangle \\ &\leq \sum_{i=1}^{\infty} \langle \gamma_i A_i(u_{\alpha_k}), u_{\alpha_k} - y\rangle \\ &\leq \alpha_k \langle u_{\alpha_k}, y - u_{\alpha_k}\rangle \\ &\leq \alpha_k \langle y, y - u_{\alpha_k}\rangle \\ &\leq 2\alpha_k \|y\|^2. \end{aligned}$$

Therefore,

$$\lim_{k\to\infty} \|A_l(u_{\alpha_k})\| = 0.$$

By virtue of Lemma 2.3, we have $u^* \in F(T_l)$. Since each set $F(T_i)$ is closed convex [16], $\mathcal{F}$ is a closed convex, too. It has only one element with the minimal norm, Thus, all the sequence $\{u_\alpha\}$ converges weakly to u^* as $\alpha \to 0$. Using this fact, (2.3) with y replaced by u^* and the property of the Hilbert space H, we can confirm that all the sequence $\{u_\alpha\}$ converges strongly to u^* as $\alpha \to 0$.
(iii) By virtue of (2.4) and the monotone property of $\mathcal{B}$ we obtain

$$\alpha\langle u_\alpha, u_\beta - u_\alpha\rangle + \beta\langle u_\beta, u_\alpha - u_\beta\rangle \geq 0$$

or

$$\|u_\alpha - u_\beta\| \leq \frac{|\alpha-\beta|}{\alpha}\|u_\beta\| \leq \frac{|\alpha-\beta|}{\alpha}\|u^*\|,$$

for each $\alpha, \beta > 0$. Theorem is proved.

Remark. Obviously, if $u_{\alpha_k} \to \tilde{u}$, where u_{α_k} is the solution of (1.6) with $\alpha = \alpha_k \to 0$, as $k \to +\infty$, then $\mathcal{F} \neq \emptyset$.

We have the following result.

Theorem 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{T_i\}_{i=1}^{\infty}$ be a sequence of non-self λ_i-strictly pseudocontractive mappings from C into H such that $\mathcal{F} := \cap_{i=1}^{\infty} \neq \emptyset$. Assume that $\{\gamma_i\}$ satisfies condition (1.7) and the functional φ is proper, convex, and Gateaux differentiable on H, and its Gateaux differentiable φ' is strongly monotone and Lipschitz continuous. Then, for every $n \geq 0$, there exists a unique solution z_{n+1} to (1.8). Moreover, if Assumption A holds, then*

$$\lim_{n\to\infty} z_n = u^* \in \mathcal{F}.$$

Proof. We know that the minimization problem (1.8) is equivalent to the following variational inequality: find $z_{n+1} \in C$ such that

$$\langle \varphi'(z_{n+1}) + \varepsilon_n(\mathcal{B}(z_n) + \alpha_n z_n) - \varphi'(z_n), v - z_{n+1}\rangle \geq 0 \quad \forall v \in C. \tag{2.5}$$

The existence and uniqueness of z_{n+1} are guaranteed by the strongly monotone property of φ'. In view of the obvious triangle inequality

$$\|z_{n+1} - u^*\| \leq \|z_{n+1} - u_{\alpha_n}\| + \|u_{\alpha_n} - u^*\|,$$

where u_{α_n} is the solution of (1.6) with $\alpha = \alpha_n$, and $\alpha_n \to 0$, it is sufficient to show that

$$\lim_{n\to\infty} \|z_{n+1} - u_{\alpha_n}\| = 0.$$

For this purpose, we introduce the functional

$$\Phi(u, z) = \varphi(u) - \varphi(z) - \langle \varphi'(z), u - z\rangle$$

where u and z play the roles for the u_{α_n} and z_n, respectively.

Since φ' is strongly monotone and Lipschitz continuous,

$$\varphi(u) - \varphi(z) \geq \langle \varphi'(z), u - z\rangle + \frac{m}{2}\|u - z\|^2 \tag{2.6}$$

and

$$\varphi(u) - \varphi(z) \leq \langle \varphi'(z), u - z\rangle + \frac{M}{2}\|u - z\|^2 \tag{2.7}$$

where m and M are the modulus of strong monotonicity and Lipschitz continuity for φ', respectively.

In light of (2.6) and (2.7), the functional Φ satisfies

$$\frac{m}{2}\|u - z\|^2 \leq \Phi(u, z) \leq \frac{M}{2}\|u - z\|^2. \tag{2.8}$$

We make the following notation:

$$\Delta_n = \|z_n - u_{\alpha_{n-1}}\|.$$

We first show that the sequence $\{\Delta_n\}_{n=0}^{\infty}$ is bounded. For this purpose, let us analyze the difference

$$\begin{aligned}
\Phi(u_{\alpha_{n-1}}, z_n) - \Phi(u_{\alpha_n}, z_{n+1}) &= \{\varphi(u_{\alpha_{n-1}}) - \varphi(z_n) - \langle \varphi'(z_n), u_{\alpha_{n-1}} - z_n\rangle\} \\
&\quad - \{\varphi(u_{\alpha_n}) - \varphi(z_{n+1}) - \langle \varphi'(z_{n+1}), u_{\alpha_n} - z_{n+1}\rangle\} \\
&= \varphi(u_{\alpha_{n-1}}) - \varphi(u_{\alpha_n}) + \langle \varphi'(z_{n+1}), u_{\alpha_n} - z_{n+1}\rangle \\
&\quad + \varphi(z_{n+1}) - \varphi(z_n) - \langle \varphi'(z_n), u_{\alpha_{n-1}} - z_n\rangle \\
&= \varphi(u_{\alpha_{n-1}}) - \varphi(u_{\alpha_n}) + \langle \varphi'(z_{n+1}), u_{\alpha_n} - z_{n+1}\rangle \\
&\quad + \varphi(z_{n+1}) - \varphi(z_n) - \langle \varphi'(z_n), z_{n+1} - z_n\rangle \\
&\quad - \langle \varphi'(z_n), u_{\alpha_{n-1}} - z_{n+1}\rangle.
\end{aligned}$$

Now making use of (2.6) and (2.7) to the above equality, we obtain

$$\begin{aligned}\Phi(u_{\alpha_{n-1}}, z_n) - \Phi(u_{\alpha_n}, z_{n+1}) \geq & \frac{m}{2}\|z_n - z_{n+1}\|^2 - \frac{M}{2}\|u_{\alpha_{n-1}} - u_{\alpha_n}\|^2 \\ & + \langle \varphi'(u_{\alpha_{n-1}}) - \varphi'(z_n), u_{\alpha_{n-1}} - u_{\alpha_n}\rangle \\ & + \langle \varphi'(z_{n+1}) - \varphi'(z_n), u_{\alpha_n} - z_{n+1}\rangle.\end{aligned} \tag{2.9}$$

On the other hand, setting $v = z_{n+1}$ and $\alpha = \alpha_n$ in (1.6), we obtain

$$\langle \mathcal{B}(u_{\alpha_n}) + \alpha_n u_{\alpha_n}, z_{n+1} - u_{\alpha_n}\rangle \geq 0,$$

and setting $v = u_{\alpha_n}$ in (2.5), we obtain

$$\langle \varphi'(z_{n+1}) + \varepsilon_n(\mathcal{B}(z_n) + \alpha_n z_n) - \varphi'(z_n), u_{\alpha_n} - z_{n+1}\rangle \geq 0.$$

We sum up the above two inequalities to obtain

$$\begin{aligned}\langle \varphi'(z_{n+1}) - \varphi'(z_n), u_{\alpha_n} - z_{n+1}\rangle \geq & \varepsilon_n\langle \mathcal{B}(u_{\alpha_n}) + \alpha_n u_{\alpha_n}, u_{\alpha_n} - z_{n+1}\rangle \\ & - \varepsilon_n\langle \mathcal{B}(z_n) + \alpha_n z_n, u_{\alpha_n} - z_{n+1}\rangle.\end{aligned}$$

Let $L = \gamma + \alpha_0$. Then, for all $x_1, x_2 \in C$, we have the following estimate

$$\|(\mathcal{B}(x_1) + \alpha_n x_1) - (\mathcal{B}(x_2) + \alpha_n x_2)\| \leq L\|x_1 - x_2\|.$$

Now, combining (2.6) and (2.7), we obtain

$$\Phi(u_{\alpha_{n-1}}, z_n) - \Phi(u_{\alpha_n}, z_{n+1}) \geq E_1 + E_2 + E_3 + E_4, \tag{2.10}$$

where

$$\begin{aligned}E_1 =& \varepsilon_n\langle (\mathcal{B}(z_n) + \alpha_n z_n) - (\mathcal{B}(u_{\alpha_n}) + \alpha_n u_{\alpha_n}), z_{n+1} - z_n\rangle + \frac{m}{2}\|z_n - z_{n+1}\|^2 \\ =& \varepsilon_n\langle (\mathcal{B}(z_n) + \alpha_n z_n) - (\mathcal{B}(u_{\alpha_{n-1}}) + \alpha_n u_{\alpha_{n-1}}), z_{n+1} - z_n\rangle \\ & + \frac{m}{2}\|z_n - z_{n+1}\|^2 \\ & + \varepsilon_n\langle (\mathcal{B}(u_{\alpha_{n-1}}) + \alpha_n u_{\alpha_{n-1}}) - (\mathcal{B}(u_{\alpha_n}) + \alpha_n u_{\alpha_n}), z_{n+1} - z_n\rangle \\ \geq & \frac{m}{2}\|z_n - z_{n+1}\|^2 - \frac{\varepsilon_n^2 L^2}{m}\|z_n - u_{\alpha_{n-1}}\|^2 - \frac{m}{4}\|z_n - z_{n+1}\|^2 \\ & - \frac{\varepsilon_n^2 L^2}{m}\|u_{\alpha_n} - u_{\alpha_{n-1}}\|^2 - \frac{m}{4}\|z_n - z_{n+1}\|^2 \\ \geq & - \frac{\varepsilon_n^2 L^2}{m}\|z_n - u_{\alpha_{n-1}}\|^2 - \frac{L^2}{m}\|u_{\alpha_n} - u_{\alpha_{n-1}}\|^2;\end{aligned}$$

$$\begin{aligned}E_2 =& \varepsilon_n\langle (\mathcal{B}(z_n) + \alpha_n z_n) - (\mathcal{B}(u_{\alpha_n}) + \alpha_n u_{\alpha_n}), z_n - u_{\alpha_{n-1}}\rangle \\ =& \varepsilon_n\langle (\mathcal{B}(z_n) + \alpha_n z_n) - (\mathcal{B}(u_{\alpha_{n-1}}) + \alpha_n u_{\alpha_{n-1}}), z_n - u_{\alpha_{n-1}}\rangle \\ & + \varepsilon_n\langle (\mathcal{B}(u_{\alpha_{n-1}}) + \alpha_n u_{\alpha_{n-1}}) - (\mathcal{B}(u_{\alpha_n}) + \alpha_n u_{\alpha_n}), z_n - u_{\alpha_{n-1}}\rangle \\ \geq & r\varepsilon_n\alpha_n\|z_n - u_{\alpha_{n-1}}\|^2 - \frac{\varepsilon_n L^2}{M}\|z_n - u_{\alpha_{n-1}}\|^2 \\ & - \frac{M}{4}\|u_{\alpha_n} - u_{\alpha_{n-1}}\|^2; 0 < r \leq 1;\end{aligned}$$

$$\begin{aligned}E_3 =&\varepsilon_n\langle(\mathcal{B}(z_n)+\alpha_n z_n)-(\mathcal{B}(u_{\alpha_{n-1}})+\alpha_n u_{\alpha_{n-1}}), u_{\alpha_n}-u_{\alpha_{n-1}}\rangle\\
&-\frac{M}{2}\|u_{\alpha_n}-u_{\alpha_{n-1}}\|^2\\
=&\varepsilon_n\langle(\mathcal{B}(z_n)+\alpha_n z_n)-(\mathcal{B}(u_{\alpha_{n-1}})+\alpha_n u_{\alpha_{n-1}}), z_n-u_{\alpha_{n-1}}\rangle\\
&+\varepsilon_n\langle(\mathcal{B}(u_{\alpha_{n-1}})+\alpha_n u_{\alpha_{n-1}})-(\mathcal{B}(u_{\alpha_n})+\alpha_n u_{\alpha_n}), u_{\alpha_n}-u_{\alpha_{n-1}}\rangle\\
&-\frac{M}{2}\|u_{\alpha_n}-u_{\alpha_{n-1}}\|^2\\
\geq& r\varepsilon_n\alpha_n\|u_{\alpha_n}-u_{\alpha_{n-1}}\|^2-\frac{3M}{4}\|u_{\alpha_n}-u_{\alpha_{n-1}}\|^2-\frac{L^2\varepsilon_n^2}{M}\|z_n-u_{\alpha_{n-1}}\|^2;\end{aligned}$$

$$\begin{aligned}E_4 &= \langle\varphi'(u_{\alpha_{n-1}})-\varphi'(z_n), u_{\alpha_n}-u_{\alpha_{n-1}}\rangle\\
&\geq -M\|u_{\alpha_{n-1}}-z_n\|\|u_{\alpha_n}-u_{\alpha_{n-1}}\|\\
&\geq -c\varepsilon_n\alpha_n\|u_{\alpha_{n-1}}-z_n\|^2-\frac{M^2\|u_{\alpha_n}-u_{\alpha_{n-1}}\|^2}{4c\varepsilon_n\alpha_n};\\
\theta &= r-c>0.\end{aligned}$$

Plugging the estimates for E_1, E_2, E_3, and E_4 in (2.10), we obtain

$$\begin{aligned}\Phi(u_{\alpha_{n-1}}, z_n)-\Phi(u_{\alpha_n}, z_{n+1}) \geq& \theta\varepsilon_n\alpha_n\|u_{\alpha_{n-1}}-z_n\|^2\\
&-\frac{(M+2m)L^2\varepsilon_n^2}{mM}\|u_{\alpha_{n-1}}-z_n\|^2-\frac{Mm+L^2}{m}\|u_{\alpha_n}-u_{\alpha_{n-1}}\|^2\\
&+r\varepsilon_n\alpha_n\|u_{\alpha_n}-u_{\alpha_{n-1}}\|^2\\
\geq&\theta\varepsilon_n\alpha_n\|u_{\alpha_{n-1}}-z_n\|^2-\frac{(M+2m)L^2\varepsilon_n^2}{mM}\|u_{\alpha_{n-1}}-z_n\|^2\\
&-\frac{(mM+L^2)^2}{4rm^2}\frac{\|u_{\alpha_n}-u_{\alpha_{n-1}}\|^2}{\varepsilon_n\alpha_n}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Phi(u_{\alpha_n}, z_{n+1}) \leq&\Phi(u_{\alpha_{n-1}}, z_n)+\Big[-\theta\varepsilon_n\alpha_n\|u_{\alpha_{n-1}}-z_n\|^2\\
&+c_1\varepsilon_n^2\|u_{\alpha_{n-1}}-z_n\|^2+c_2\frac{\|u_{\alpha_n}-u_{\alpha_{n-1}}\|^2}{\varepsilon_n\alpha_n}\Big],\end{aligned}$$

where $c_1 = (M+2m)L^2/mM$ and $c_2 = (mM+L^2)^2/4rm^2$.

Considering analogue of the above inequality from $n = 0$ to N, summing them side-by-side and using (2.9), we obtain

$$\begin{aligned}\left(\frac{m}{2}\right)\Delta_{n+1}^2 \leq&\left(\frac{M}{2}\right)\Delta_1^2\\
&+\sum_{n=1}^{N}\left[-\theta\varepsilon_n\alpha_n\Delta_n^2+c_1\varepsilon_n^2\Delta_n^2+c_2\left(\frac{\alpha_n-\alpha_{n+1}}{\alpha_n}\right)^2\|u^*\|^2(\varepsilon_n\alpha_n)^{-1}\right].\end{aligned}$$

The above inequality, in view of Assumption A and Lemma 2.5 of [10] confirms the boundedness of the sequence $\{\Delta_n\}_{n=1}^{\infty}$. Moreover, the last inequality in view of the boundedness of the sequence $\{\Delta_n\}_{n=1}^{\infty}$ gives the following estimate

$$\sum_{n=1}^{\infty} \theta\varepsilon_n\alpha_n\Delta_n^2 < \infty.$$

This, in view of the divergence of the series$\sum_{n=1}^{\infty} \varepsilon_n\alpha_n$ and the above observations, confirms that

$$\lim_{n\to\infty} \Delta_n = 0.$$

This completes the proof.

EXAMPLE. Let $1/2 < k_1 < 1, k_2 > 0, k_1 + k_2 < 1$. Then, for the sequences of the form

$$\varepsilon_n = (1+n)^{-k_1}, \quad \text{and} \quad \alpha_n = (1+n)^{-k_2},$$

Assumption A is fullfied.

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