Int. Journal of Math. Analysis, Vol. 3, 2009, no. 12, 603 - 610

On the Convergence of Ishikawa Iterates Defined by Nonlinear Quasi-Contractions

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Abstract

In [8] Lj. B. Ćirić proved general result on the convergence of Ishikawa iterates of nonlinear quasi - contractions defined on Takahashi convex metric space. In this paper we present two generalizations of Ćirić's result.

Mathematics Subject Classification: 54H25, 47H10

Keywords: fixed point, nonlinear quasi contraction, Ishikawa iterates

1 Introduction

Let X be a nonempty set and $f : X \to X$ arbitrary mapping. $x \in X$ is a fixed point for f if $x = f(x)$. If $x_0 \in X$, we say that a sequence (x_n) defined by $x_n = f^n(x_0)$ is a sequence of Picard iterates of f at point x_0 or that (x_n) is the orbit of f at point x_0.

Let (α_n) and (β_n) be two sequences of real numbers. Let (X, d) be a Takahashi convex metric space and W Takahashi convex structure on X. Then for arbitrary $x_0 \in X$ sequence of Ishikawa iterates of $(f, (\alpha_n), (\beta_n))$ at point x_0 is defined by:

$$y_n = W(f(x_n), x_n, \beta_n) \quad x_{n+1} = W(f(y_n), x_n, \alpha_n).$$

Lj. Ćirić [5] first introduced the notion of quasi - contractions and proved the fixed point theorem for this class of mappings. Ćiri'c's result was extended to nonlinear quasi - contractions by A. A. Ivanov [10]. In paper [1] we present new generalizations of Ivanov's fixed point theorem.

In modern nonlinear analysis and its applications theory of Ishikawa iterates (see [9],[11],...) has important role because it can be applied in many cases in which classical methods (convergence of Picarad or Mann's iterations) is not useful. In many papers Ishikawa iteration sequence was used as tool to obtain approximative fixed points of nonexpansive and pseudo-concractive mappings defined on Hilbert and Banach spaces.

First result on the convergence of Ishikawa iterates defined by quasi - contractions was obtained by L. Qihou [13] for Hilbert spaces. This result is generalized in L. Qihou [14] for non-compact subset of a Hilbert space, C. E. Chidume [3] for Banach spaces L^p and ℓ^p $(2 < p)$, C. E. Chidume [4] for Banach spaces L^p and Xu [17] for arbitrary Banach space. B. E. Rhoades [15] proved first result on the convergence of Ishikawa iterates defined by nonlinear quasi - contractions. In [7] Lj. B. Ćirić proved that results of Xu and Rhoades is uncorrect. He proved general result for nonlinear quasi - contractions defined on Takahashi convex metric space [8]. In this paper we present two generalizations of Ćirić's result.

2 Preliminary Notes

In 1928, K. Menger [12] introduce the notion of convexity in metric space.

Definition 2.1 *Let (X, d) be a metric space, such that for each $x, y \in X$ with $x \neq y$ there exists $z \in X$ $(x \neq z \neq y)$ such that $d(x, z) + d(z, y) = d(x, y)$. Then X is called a convex metric space.*

Menger's definition includes many geometric properties of Euclidian convex sets, but it is restrictive in fixed point applications.

In 1970, W. Takahashi [16] introduced the new concept of convexity in metric space and generalized some important fixed point theorems previously proved for Banach spaces.

Definition 2.2 *Let (X, d) be a metric space and $I = [0, 1]$ the closed unit interval. A Takahashi convex structure on X is a function $W : X \times X \times I \to X$ which has the property that for every $x, y \in X$ and $\lambda \in I$*

$$d(z, W(x, y, \lambda) \leq \lambda d(z, x) + (1 - \lambda) d(z, y)$$

for every $z \in X$.

If (X, d) is equipped with a Takahashi convex structure, then X is called a Takahashi convex metric space, or a metric space of hyperbolic type.

The term "metric space of hyperbolic type" is introduced by W.A. Kirk [11] and today is widely used.

Let (X, d) be a Takahashi convex metric space and

$$m(x, y) = W(x, y, \frac{1}{2}),$$

for any $x, y \in X$. From definition follows

$$d(m(x, z), m(y, z)) \leq \frac{1}{2} d(x, y).$$

In hyperbolic geometry we have strict inequality, while in parabolic geometry it is equality. This implies that "metric space of non-elliptic type" could be more appropriate term than "metric space of hyperbolic type".

By Φ we denote the set of real functions $\varphi : [0, \infty) \to [0, \infty)$ which have the following properties:

(a) $\varphi(0) = 0$;

(b) $\varphi(r) < r$ for all $r > 0$;

(c) $\lim_{x\to\infty}(x - \varphi(x)) = \infty$.

Define

$\Phi_1 = \{\varphi \in \Phi : \ \varphi$ is monotone nondecreasing and $\overline{\lim}_{t\to r+}\varphi(t) < r$ for $r > 0\}$,

$\Phi_2 = \{\varphi \in \Phi : \overline{\lim}_{t\to r}\varphi(t) < r$ for any $r > 0\}$.

3 Main Results

In [8] Lj. B. Ćirić proved the following result:

Theorem 3.1 *(Lj. B. Ćirić [8]) Let (X, d) be a Takahashi convex complete metric space, $f : X \to X$ and (α_n), (β_n) two sequences of real numbers such that $0 < \alpha_n, \beta_n < 1$ for all n and $\sum \alpha_n = \infty$. If there exists $\varphi \in \Phi_1$ such that*

$$\begin{aligned} d(f(x), f(y)) \ \leq \ & \max\{\varphi(d(x, y)), \varphi(d(x, f(x))), \varphi(d(y, f(y))), \\ & \varphi(d(x, f(y))), \varphi(d(f(x), y))\}, \end{aligned}$$

for any $x, y \in X$, then for arbitrary $x_0 \in X$ sequence of Ishikawa iterates of $(f, (\alpha_n), (\beta_n))$ at point x_0 is defined by:

$$y_n = W(f(x_n), x_n, \beta_n) \quad x_{n+1} = W(f(y_n), x_n, \alpha_n)$$

converge to unique fixed point of f.

Existence and uniqueness of fixed point also follows from our Theorem 1 [1]. We can see that hypotheses for the gauge function φ in Theorem 1 are stronger than assumptions of Boyd - Wong's [2] fixed point theorem.

We can see that for Theorem 1 assumptions

$$\lim_{t\to\infty}(t-\varphi(t))=\infty$$

and φ is monotone nondecreasing, were needed for existence of fixed point.

Example 3.2 - *Ćirić [6]. Let* $X=(0,+\infty)$ *with usual metric,* $\varphi:[0,+\infty)\to[0,+\infty)$ *mapping defined by:*

$$\varphi(t)=\begin{cases} \frac{t}{2}, & t\in[0,1); \\ t-\frac{1}{2}, & t\in[1,+\infty) \end{cases},$$

and $f(x)=x+1$.

If $x<y$ then:

$$d(f(x),f(y))=d(x,y)<(d(x,y)+1)-\frac{1}{2}=\varphi(d(x,f(y))).$$

φ is continuous, monotone nondecreasing and $\varphi(t)<t$, but

$$\lim_{t\to\infty}(t-\varphi(t))\neq\infty$$

and f is fixed point free.

Example 3.3 - *Ćirić [6]. Let* $X=(0,+\infty)$,

$$d(x,y)=\frac{|x-y|}{1+|x-y|},$$

$\varphi:[0,+\infty)\to[0,+\infty)$ *mapping defined by:*

$$\varphi(t)=\begin{cases} \frac{t}{2}, & t\in[0,\frac{1}{2})\cup[1,+\infty); \\ t-(1-t)^2, & t\in[\frac{1}{2},1), \end{cases},$$

and $f(x)=x+1$.

Then φ is not monotone nondecreasing and all other conditions of Theorem 1 are satisfied. f is also fixed point free.

Now we need the following Lemma.

Lemma 3.4 *Let* $\varphi\in\Phi_2$. *Then there exists* $\psi\in\Phi_1$ *such that*

$$\varphi(x)\leq\psi(x)<x,$$

for each $x>0$.

Proof: Let real function $\psi : [0, \infty) \to [0, \infty)$ be defined by

$$\psi(x) = \sup_{t \in [0,x]} \varphi(t).$$

Then ψ is monotone nondecreasing, $\psi(0) = 0$,
$\overline{\lim}_{t \to x+} \psi(t) < x$ and $\varphi(x) \leq \psi(x) < x$ for any $x > 0$.

If φ is bounded then ψ is bounded, which implies that

$$\lim_{x \to \infty} (x - \psi(x)) = \infty.$$

Suppose that

$$\lim_{x \to \infty} \psi(x) = \infty.$$

This implies that

$$\overline{\lim_{x \to \infty}} \varphi(x) = \infty.$$

Then for arbitrary $\varepsilon > 0$ there exists real function $g : [0, \infty) \to [0, \infty)$ such that:

$$g(x) \in \{t \in [0, x] : \psi(x) - \varphi(t) < \varepsilon\}.$$

Now, for any $R > 0$ there exists $x > 0$ such that

$$x > \varphi(x) > R + \varepsilon,$$

which implies that $g(x) > R$. So

$$\lim_{x \to \infty} g(x) = \infty.$$

Then

$$\begin{aligned}\underline{\lim}_{x \to \infty} (x - \psi(x)) &= \underline{\lim}_{x \to \infty} (((x - g(x)) + (g(x) - \varphi(g(x))) + (\varphi(g(x)) - \psi(x))) \geq \\ &\geq \underline{\lim}_{x \to \infty} ((x - g(x)) + \underline{\lim}_{x \to \infty} (g(x) - \varphi(g(x))) + \underline{\lim}_{x \to \infty} (\varphi(g(x) - \psi(x))) = \infty,\end{aligned}$$

because

$$\underline{\lim}_{x \to \infty} (g(x) - \varphi(g(x))) = \infty.$$

So

$$\lim_{x \to \infty} (x - \psi(x)) = \infty$$

which implies that $\psi \in \Phi_1$.$\diamondsuit$

Now we shall prove our next result.

Theorem 3.5 *Let (X, d) be a Takahashi convex complete metric space, $f : X \to X$ and (α_n), (β_n) two sequences of real numbers such that $0 < \alpha_n, \beta_n < 1$ for all n and $\sum \alpha_n = \infty$. If there exists $\varphi \in \Phi_2$ such that*

$$\begin{aligned} d(f(x), f(y)) \leq \; & \max\{\varphi(d(x, y)), \varphi(d(x, f(x))), \varphi(d(y, f(y))), \\ & \varphi(d(x, f(y))), \varphi(d(f(x), y))\}, \end{aligned}$$

for any $x, y \in X$, then for arbitrary $x_0 \in X$ sequence of Ishikawa iterates of $(f, (\alpha_n), (\beta_n))$ at point x_0 is defined by:

$$y_n = W(f(x_n), x_n, \beta_n) \quad x_{n+1} = W(f(y_n), x_n, \alpha_n)$$

converge to unique fixed point of f.

Proof: From Lemma 3.4 follows that there exists $\psi \in \Phi_1$ such that:

$$\begin{aligned} d(f(x), f(y)) \leq \; & \max\{\psi(d(x, y)), \psi(d(x, f(x))), \psi(d(y, f(y))), \\ & \psi(d(x, f(y))), \psi(d(f(x), y))\}. \end{aligned}$$

Therefore, the hypotheses of Theorem 3.1 are satisfied. $\diamondsuit$

Lemma 3.6 *Let $\varphi_1, \ldots, \varphi_n \in \Phi_1$. Then there exists $\psi \in \Phi_1$ such that*

$$\varphi_k(x) \leq \psi(x) < x,$$

for each $1 \leq k \leq n$ and $x > 0$.

Proof:
Let

$$\psi(x) = \max\{\varphi_1, \ldots, \varphi_n\}.$$

Then it is easy to show that: $\psi(0) = 0$, ψ is monotone nondecreasing, $\varphi_k(t) \leq \psi(t) < t$ $(1 \leq k \leq n)$ for all $t > 0$ and

$$\overline{\lim_{t \to x+}} \psi(t) < x.$$

Now we shall prove that $\lim_{x\to\infty}(x - \psi(x)) = \infty$.

Let $R > 0$ be arbitrary. Then for any $1 \leq k \leq n$ there exists $S_k > 0$ such that $t - \varphi_k(t) > R$ for all $t > S_k$. Then for all

$$t > \max_{1 \leq k \leq n} S_k$$

we have

$$t - \psi(t) = t - \max_{1 \leq k \leq n} \varphi_k(t) = \min_{1 \leq k \leq n} (t - \varphi_k(t)) > R.$$

So $\lim_{x\to\infty}(x - \psi(x)) = \infty$. $\diamondsuit$

Now we shall prove our main result, which is a common generalization of Theorem 3.1 and Theorem 3.5.

Theorem 3.7 *Let (X, d) be a Takahashi convex complete metric space, $f : X \to X$ and (α_n), (β_n) two sequences of real numbers such that $0 < \alpha_n, \beta_n < 1$ for all n and $\sum \alpha_n = \infty$. If there exists $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \in \Phi_1 \cup \Phi_2$ such that*

$$\begin{aligned} d(f(x), f(y)) \quad \leq \quad & \max\{\varphi_1(d(x, y)), \varphi_2(d(x, f(x))), \varphi_3(d(y, f(y))), \\ & \varphi_4(d(x, f(y))), \varphi_5(d(f(x), y))\}, \end{aligned}$$

for any $x, y \in X$, then for arbitrary $x_0 \in X$ sequence of Ishikawa iterates of $(f, (\alpha_n), (\beta_n))$ at point x_0 is defined by:

$$y_n = W(f(x_n), x_n, \beta_n) \quad x_{n+1} = W(f(y_n), x_n, \alpha_n)$$

converge to unique fixed point of f.

Proof: Let

$$\varphi_k^*(t) = \begin{cases} \varphi_k(t), \; \varphi_k \in \Phi_1 \backslash \Phi_2; \\ \sup_{s\in[0,t]} \varphi_k(s), \; \varphi_k \in \Phi_2 \end{cases}, \; (1 \leq k \leq 5).$$

From Lemma 3.4 it follows that $\varphi_k^* \in \Phi_1$ $(1 \leq k \leq 5)$.

From Lemma 3.6 it follows that there exists real function $\varphi \in \Phi_1$ such that:

$$\varphi_k^*(x) \leq \varphi(x) < x, \; (1 \leq k \leq n) \text{ for each } x > 0, \text{ which implies}$$

$$\begin{aligned} d(f(x), f(y)) \quad \leq \quad & \max\{\varphi(d(x, y)), \varphi(d(x, f(x))), \varphi(d(y, f(y))), \\ & \varphi(d(x, f(y))), \varphi(d(f(x), y))\}. \end{aligned}$$

Therefore, the hypotheses of Theorem 3.1 are satisfied. $\diamondsuit$

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Received: September 29, 2008