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Banach Spaces and Extreme Points

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Abstract. Many properties of Banach spaces are characterized by structure and properties of extreme points on closed unit sphere of Banach spaces. In this paper we considered the connection between the set of extreme points and duality of Banach spaces.

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1. INTRODUCTION

Many properties of Banach spaces are characterized by structure and properties of extreme points on closed unit sphere of Banach spaces. It is possible to attribute to such properties: a duality of Banach spaces, an equivalence of norms in Banach spaces and a completeness of a semi-norm in Banach spaces.

In this paper we have considered some connections between set of extreme points and a duality of Banach spaces and behaviour of semi-norm of Banach spaces.

In paper [1], it is considered the question of existence of extrem points (on the basis of Theorem Krein-Milman) and the condition on set of extreme points under which the Banach space is dual. This condition is presented as in the following test:

Test 1: Let B be a real Banach space and B_1 its closed unit ball. Then B is not a dual space if B_1 has no extreme points.

In addition, is presented the Theorem characterizing property of invariance of extreme points relative to isometry:

Theorem 1. *Let A and B be real Banach spaces and $T : A \to B$ is a linear isometry. Then T preserves extreme points, namely $T(e)$ is an extreme point of B_1 if and only if e is an extreme point of A_1.*

Thus if the set of extreme points in the unit ball A_1 is closed/ open, then the set of extreme points in B_1 is also closed/ open if there is a linear isometry between A and B.

In connection with Test 1, let's consider the following problem presented in [6, V.11, Exercise 5]:

If the closed unit sphere of the infinite dimensional Banach space X contains only final number of extreme points, then, the space X is not isometrically isomorphic to the space conjugate to some Banach space.

On the one hand, it is a stronger condition on quantity of extreme points, and, on the other hand, it is weaker condition on a duality as it is not required isometric isomorphism with its own conjugate space. Clearly, if previous statement is true it will not be difficult to prove *Test 1.*

As other sources, we have used papers [2], [3], [4], [5], [7]. An extreme point of X_1 is called weak *-extreme if it continues to be an extreme point of X_1^{**}. It is known from R. R. Phelps [2] that for a compact set K any extreme point of the unit ball of $C(K)$ is weak *-extreme. Importance of these points to the geometry of a Banach space was presented in H. Rosenthal [3], where it was proved that a Banach space has the Radon-Nikodym property (RNP) if and only if for every equivalent norm the unit ball has a weak *-extreme point have served.

In [4], weak *-extreme points are considered.

Let $x_0 \in X_1$ be such that $|x^*(x_0)| = 1$ for all $x^* \in \partial_e X_1^*$.

Proposition 1. *Let X be a Banach space and let $x_0 \in X_1$ be such that $|x^*(x_0)| = 1$ for all $x^* \in \partial_e X_1^*$. Then $|\tau(x_0)| = 1$ for all $\tau \in \partial_e X_1^{**}$.*

In [5], the unitaries are considered and a condition under which a semi-norm p is a complete norm is given. In this connection we consider a condition on extreme points at which the semi-norm p is complete norm in space X.

II. The basic result: Let X be real Banach space X^* is dual to X, S_X and S_{X^*} are the unit spheres of spaces X and X^*, E_X and E_{X^*} are the sets of extreme points of spaces X and X^*, respectively. Let $C(Q)$ is a space of the real continuous functions given on compact Hausdorff space Q.

Let $u \in S_X$ and $S_u = \{x^* \mid x^*(u) = 1, \ x^* \in S_{X^*}\}$ be the state space.

Lemma 1. *Let X be a real Banach space and $u \in S_X$ be an extreme point of space X. Then, the mapping $T : X \to C(S_u)$ is isometric isomorphism.*

Proof. First of all, we should note the fact, by corollary [6, Corollary V.4.4.] every Banach space X is isomorphic to some closed subspace of Banach space $C(Q)$ of the real continuous functions given on some bicompact Hausdorff space Q, i.e. space X isometrically and isomorphically is embedded in space

$C(Q)$ for the some bicompact Hausdorff space Q. We shall take Q as set S_u, i.e. $Q = S_u$. ∎

By Theorem (Alaoglu) [6, Theorem V. 4.2.] the set S_u is weak *-compact subset of unit sphere S_{X^*} and, is convex set as well. Therefore the set S_u contains, at least, one extreme point (under Theorem Krein-Milman) in unit spheres S_{X^*}.

Let's consider mapping

$$T : X \to C(S_u),$$

where $u \in S_X$ is an extreme point. We will show that mapping T is isometric isomerphism. By lemma [6, Lemma V.8.6.] and [7] for every $x^* \in S_u$ it is possible to determine $x^*_{S_u}$ from X^* by equality

$$x^*_{S_u} f = f(x^*), \text{ where } f \in C(S_u) \tag{1.1}$$

As was noted above, the mapping T is isometric isomorphism embedding. Therefore, on the basis of the formula (1.1) it is possible to set mapping T by the equation

$$T(x)\, x^* = x^*(x).$$

It is not difficult to see that $T(u) = 1$. Now, let's show that T is isometric mapping onto. Really, let there be an element $y \in C(S_u)$ which does not exist $x \in X$ such that $T(x) = y$. As y an element of space of functions $C(S_u)$, taking into account definition of set S_u and the formula (1.1), this element should be of form $y = f\left(x^*_{S_u}\right)$ where $x^*_{S_u}$ is any element of set S_u. But $x^*_{S_u}$ is a functional determined on space X, and, hence, in space X should exists some element x on which this functional takes value $x^*_{S_u}(x)$, and function f accepts value $f\left(x^*_{S_u}(x)\right)$. But, by assumption such element x does not exist, therefore, it contradicts our assumption. Thus, T mappings space X on all space $C(S_u)$ and, hence, is mapping onto.

Let's define linear mapping

$$T^* : C^*(S_u) \to X^*$$

by equality $(T^* y^*)\, x = y^*(Tx)$. It is easy to see the following. If T is defined as above then range of T is closed, because T is one-one mapping we have (by the closed range Theorem) that mapping T^* is onto.

Before to proceed to following theorem we should present some reasonings.

Provided that Banach space X is isometrically isomorphic to Banach space $C(S_u)$, i.e. $X = C(S_u)$, by Lemma V. 8.6. [6] and [7] each extreme point of unit sphere S_{X^*} of space X^* looks like $\alpha . x^*_{S_u}$, where $|\alpha| = 1$, is an extreme point of unit sphere S_{X^*}. By Lemma V.8.7.[6] the set S_u is isomorphic and homemorphic maps in a subset $\widehat{S_u} \subset E_{X^*}$ of extreme points set of sphere S_{X^*}. Since S_u is a bicompact subset in S_{X^*} then, from here follows that, at least, set E_{X^*} contains a bicompact subset of extreme points.

Theorem 2. *Let X be a real Banach space and $u \in S_X$ be an extreme point of space X. Then $\overline{span}\,(S_u) = X^*$ $(\overline{co}\,(S_u) = X^*)$.*

Proof. First of all, we shall note, that the proof of this Theorem essentially is based on Lemma V.8.6.[6]). For the proof of the Theorem it is enough to show, that $\overline{span}\,(S_u) = S_{X^*}$ $(\overline{co}\,(S_u) = S_{X^*})$. By Lemma 1, the space X is isometrically isomorphic to Banach space $C\,(S_u)$, therefore further, without loss of a generality, we should accept that $X = C\,(S_u)$.Let K be a set of all points of space X^*, looking like $\alpha.x^*_{S_u}$, where $|\alpha| = 1$, so $K \subset S_{X^*}$.Space X^* is considered in its X^* topology. As S_{X^*} is convex and X^*-closed then by Lemma V.2.4.[6] (X^*-completeness) $\overline{co\,(K)} = \overline{co}\,(K) \subseteq S_{X^*}$. Now we let assume, that there is a point $x^* \in S_{X^*}$ which does not belong to set $\overline{co}\,(K)$. If $x^* \notin \overline{co}\,(K)$ then by Theorem V.3.10. and Corollary V.2.12. [6] we can find $x \in X$ and such real constants c and $\varepsilon > 0$ that

$$\operatorname{Re} x^*\,(x) \geq c; \;\; \operatorname{Re} \alpha.x_{S_u} \leq c - \varepsilon, \;\; |\alpha| = 1,$$

where x_{S_u}(under the assumption, that $X = C\,(S_u)$) corresponds to an element f_{S_u}. An element f_{S_u} (by Lemma V.8.6.[6]) is defined from a condition that for every $x^* \in S_u$ it is possible to determine $x^*_{X_u}$ by equality

$$x^*_{S_u}\,(f) = f\,(x^*) = f_{S_u}, \text{ where } f, \; f_{S_u} \in C\,(S_u)\,.$$

Then $|x_{S_u}| \leq c - \varepsilon$ and, hence $|x^*| \geq 1$. But it contradicts the fact that $x^* \in S_{X^*}$. Thus $\overline{co}\,(K) \supset S_X$ that is $\overline{co}\,(K) = S_{X^*}$. ∎

Now we apply the Lemma 1 and the Theorem 1 to the proof of strengthened variant *Test 1 and to the proof of completeness of a semi-norm. But before we prove the following statement which was mentioned in introduction.*

Statement: If the closed unit sphere infinite dimensional Banach space X contains only final number of extreme points then space X is not isometrically isomorphic to space Y^*, conjugate to the some Banach space Y.

Proof. Let $K = \{u_1, u_2, ..., u_n\} \subset S_X$ is a set from n extreme points of unit sphere S_X of space X. Let Y is another Banach space and Y^* is the space conjugate to Y. From the proof of the Lemma 1 and Theorem 1 follows that unit sphere S_{Y^*} of space Y^* contains bicompact subset E_{Y^*} of extreme points. But by a condition of the Statement space X is isometrically isomorphic, then by the theorem 3.1[1], the set K of extreme points of space X should be isometrically and isomorphic mapped on set E_{Y^*} of extreme points of space Y^* and on the contrary. But K is final set and E_{Y^*} is infinite bicompact set. Hence, we come to the contradiction. ∎

On the basis of this Statement it is possible to formulate the following Theorem:

Theorem 3. *The infinite dimensional Banach space X is isometrically isomorphic to space Y^*, conjugate to the some Banach space Y if unit sphere S_X of space X contain infinite bicompact subset $E_X \subset S_X$ of extreme points.*

Proof. The proof follows from the previous proposition. ∎

The following Theorem shows connection between extreme points and semi-norms.

Let us $u \in S_X$, then the state space $S_u = \{x^* \mid x^*(u) = 1,\ x^* \in S_{X^*}\}$.

Let' s define a semi-norm p on X as $p(x) = \sup\{x^*(x) \mid \text{for } x^* \in S_u\}$.

Theorem 4. *If $u \in S_X$ is an extreme point of space X then p is complete norm on the Banach space X.*

Proof. Let $u \in S_X$ be an extreme point on sphere S_X. If $p(x) = 0$ then as $x^*(x) = 0$ for all $x^* \in S_u$ and, hence, for all $x^* \in X^*$. We get, that $x = 0$. It is clear that $p(x) \leq \|x\|$. By Lemma 1, the mapping $T : X \to C(S_u)$ is isometric isomorphism which is given as $T(x)x^* = x^*(x)$ for $x \in X$ and $x^* \in S_u$. In addition $T(u) = 1$. Therefore mapping T^* is onto. Hence, by the closed range theorem, range of T is closed. Thus p is a complete norm. ∎

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