Positive Solutions for Boundary Value Problems on a Half Line

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Abstract

In this paper, we consider the multiplicity of positive solutions for one-dimensional $p$-Laplacian differential equation on the half line. By using fixed point theorem due to Avery and Peterson, we provide sufficient conditions for the existence of at least three positive solutions.

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1 Introduction

In this paper, we study the existence of triple positive solutions to the boundary value problem (BVP) for the one-dimensional $p$-Laplacian on a half line

\[
\begin{aligned}
\phi_p(x'(t))' + h(t)f(t, x(t), x'(t)) &= 0, \quad 0 < t < \infty, \\
x(0) &= 0, \quad \lim_{t \to +\infty} x'(t) = 0,
\end{aligned}
\]  

(1.1)

where $\phi_p(s) = |s|^{p-2} s, \ p > 1$.

The study of multi-point boundary value problems for linear second-order ordinary differential equation was initiated by Il'in and Moiseev [1, 2]. Since then there has been much current attention focused on the study of nonlinear multi-point boundary value problems [3, 6, 7] and the references therein. Eq. (1.1) with $0 < t < +\infty$ substituted by $0 < t < 1$, and sometimes with the nonlinear term $f$ without the first-order derivative have been studied by several researchers, see [5]. Especially, the study of triple positive solutions attracts much attention.

In [4], the authors studied the problem

\[
\begin{aligned}
x'' + \phi(t)f(t, x, x') &= 0, \quad 0 < t < +\infty, \\
x(0) &= 0, \quad \lim_{t \to +\infty} x'(t) = 0.
\end{aligned}
\]  

In [7], the author investigated the problem
\[
\begin{align*}
&x'' - k^2 x(t) + f(t, x(t)) = 0, \quad 0 < t < +\infty, \\
x(0) = 0, \quad \lim_{t \to +\infty} x(t) = 0
\end{align*}
\]

Motivated by the above works, we investigated problem (1.1). The interesting point here is that when \( \alpha \neq 0 \), problem (1.1) is a multi-point boundary value problem on infinite interval with \( p \)-Laplacian, which has never been studied before. What’s more, we at least three positive solution. And when \( \alpha = 0 \), although the problem become the same as in [9] when \( \beta = 0 \), our process of getting the main results is different. In section five, we give an example to illustrate our main results.

Throughout this paper, we always suppose that \( h, f \) satisfy

\begin{itemize}
  \item [(H_1)] \( h \in C(R_+, R_+), h \) is not identically zero on any subinterval of \((0, \infty)\) and \( \int_0^{+\infty} h(s) ds < +\infty, \int_0^{+\infty} \phi_q \left( \int_0^{+\infty} h(s) ds \right) d\tau < +\infty. \)
  \item [(H_2)] \( f(t, (1 + t)u, v) \in C(R_+^3, R_+), f(t, 0, 0) \) is not identically zero on any subinterval of \((0, +\infty)\) and when \( u, v \) are bounded \( f(t, (1 + t)u, v) \) is bounded on \([0, +\infty)\).
\end{itemize}

For convenience, here we set \( \phi_q(s) = |s|^{q-1} s, \frac{1}{p} + \frac{1}{q} = 1 \) is the inverse function to \( \phi_p(s) \).

\section{Preliminary Notes}

For the convenience of the reader, we will present some definitions in this section which are important in the proof process of the main results.

Let
\[
X = \{ x \in C^1[0, \infty), \sup_{0 \leq t < +\infty} \frac{x(t)}{1 + t} < +\infty, \lim_{t \to +\infty} x'(t) = 0 \}\quad (2.1)
\]

with the norm \( \| x \| = \max \{ \| x \|, \| x' \|_\infty \} \), where \( \| x \|_1 = \sup |\frac{x(t)}{1 + t}|, \| x' \|_\infty = \sup_{0 \leq t < +\infty} |x'(t)| \). It is clearly that \((X, \| \cdot \|)\) is a Banach space.

Define the cone \( P \subset X \) by
\[
P = \{ x \in X, x(t) \geq 0, t \in [0, +\infty), x(0) = 0, x \) is concave on\([0, +\infty)\} \}
\]

\begin{definition}
A map \( \alpha \) is said to be a nonnegative continuous \textbf{concave} functional on \( P \) provided that \( \alpha : P \to [0, \infty) \) is continuous and
\[
\alpha(\lambda x + (1 - \lambda)y) \geq \lambda \alpha(x) + (1 - \lambda)\alpha(y).
\]
\end{definition}
Respectively, a map $\beta$ is said to be a nonnegative continuous convex functional on $P$ provided that $\beta : P \to [0, \infty)$ is continuous and

$$\beta(\lambda x + (1 - \lambda)y) \leq \lambda\beta(x) + (1 - \lambda)\beta(y)$$

for all $x, y \in P$ and $0 \leq \lambda \leq 1$.

Let $\alpha$, $\gamma$, $\theta$, $\psi$ be nonnegative continuous maps on $P$ with $\alpha$ concave, and $\theta$, $\gamma$ convex. Then for positive numbers $a$, $b$, $c$, $d$ we define the following subsets of $P$

$$P(\gamma, d) = \{ x \in P \mid \gamma(x) < d \},$$
$$P(\alpha, b, \gamma, d) = \{ x \in \overline{P(\gamma, d)} \mid \alpha(x) \geq b \},$$
$$P(\alpha, b, \theta, c, \gamma, d) = \{ x \in \overline{P(\gamma, d)} \mid \alpha(x) \geq b, \theta(x) \leq c \},$$
$$R(\psi, a, \gamma, d) = \{ x \in \overline{P(\gamma, d)} \mid \psi(x) \geq a \},$$

then it is obvious that $P(\gamma, d)$, $P(\alpha, b, \gamma, d)$ and $P(\alpha, b, \theta, c, \gamma, d)$ are convex and $R(\psi, a, \gamma, d)$ are closed.

Next we state Avery-Peterson fixed point theorem.

**Theorem 2.2** ([3]) Let $P$ be a cone in Banach space $E$. Let $\gamma$ and $\theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be nonnegative continuous concave functional on $P$, and $\psi$ be a nonnegative continuous functional on $P$ satisfying

$$\psi(\lambda x) \leq \lambda \psi(x) \text{ for all } 0 \leq \lambda \leq 1.$$  

and

$$\alpha(x) \leq \psi(x), \quad \|x\| \leq M\gamma(x) \text{ for all } x \in \overline{P(\gamma, d)}.$$  

with $M, d$ be positive numbers. Suppose that $T : P \to P$ is completely continuous and there exist positive numbers $a$, $b$, $c$, $d$ with $a < b$ such that

$$(S_1) \text{ \{ } x \in P(\alpha, b, \theta, c, \gamma, d) \mid \alpha(x) > b \text{ \} \neq \emptyset \text{ and } \alpha(Tx) > b \text{ for } x \in P(\alpha, b, \theta, c, \gamma, d);}$$

$$(S_2) \text{ } \alpha(Tx) > b \text{ for } x \in P(\alpha, b, \gamma, d) \text{ with } \theta(Tx) > c;$$

$$(S_3) 0 \notin R(\psi, a, \gamma, d) \text{ and } \psi(Tx) < a \text{ for } x \in R(\psi, a, \gamma, d) \text{ with } \psi(x) = a.$$  

Then $T$ has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(x_i) \leq d, \quad i = 1, 2, 3; \quad \psi(x_1) < a; \quad \psi(x_2) > a \text{ with } \alpha(x_2) < b; \quad \alpha(x_3) > b.$$
3 Some Lemmas

Let $k > 1$ be a constant and we define the nonnegative continuous functionals $\alpha, \gamma, \theta, \psi$ on $P$ by
\[
\begin{align*}
\alpha(x) &= \frac{k}{k+1} \min_{1 \leq t \leq k} x(t), \quad \gamma(x) = \sup_{0 \leq t < +\infty} x'(t), \\
\psi(x) &= \theta(x) = \frac{x(0)}{1+\epsilon} \quad \text{for } x \in P.
\end{align*}
\] (3.1)

**Lemma 3.1** For $x \in P$, $\|x\|_1 \leq \|x'\|_\infty$.

**Proof.** For any $x \in P$, we have $x(t) = \int_0^t x'(s)ds \leq t \|x'\|_\infty$, $0 \leq t \leq +\infty$.

then we have $\frac{x(t)}{1+t} \leq \frac{t}{1+t} \|x'\|_\infty$. So we can get $\|x\|_1 \leq \|x'\|_\infty$. The proof is complete.

**Lemma 3.2** For $x \in P$, $\alpha(x) \geq \frac{1}{k+1} \theta(x)$.

**Proof.** It can be easily seen that for any $x \in P$, $x$ is increasing on $[0, +\infty)$. Meanwhile, noticing $x'(t) = 0$, the function $\frac{x(t)}{1+t}$ achieve its maximum at $\sigma \in [0, +\infty)$, then $\theta(x) = \frac{x(\sigma)}{1+\sigma}$. In fact, if $\theta(x) = \lim_{t \to +\infty} \frac{x(t)}{1+t}$, then $\lim_{t \to +\infty} \frac{x(t)}{1+t} = 0$, it is a contradiction. Furthermore, $x$ is concave, so
\[
\alpha(x) = \frac{k}{k+1} x\left(\frac{1}{k}\right) = \frac{k}{k+1} \left(\frac{k-1+k\sigma}{k+k\sigma} \cdot \frac{1}{k-1+k\sigma} + \frac{1}{k+k\sigma}\right)
\]
\[
\geq \frac{1}{k+1} \cdot \frac{x(\sigma)}{1+\sigma} = \frac{1}{k+1} \theta(x).
\]

Then the proof is completed.

Define the operator $T : P \to C^1[0, +\infty)$ by
\[
(Tx)(t) = \int_0^t \left(\phi_q \int_{\tau}^{+\infty} h(s)f(s, x(s), x'(s))ds\right)d\tau.
\] (3.2)

Since the Arzela-Ascoli theorem fails to work in the space $X$, we need a modified compactness criterion to prove $T$ is compact. In the following, we will present an explicit one. For more general cases, we refer the readers to [7,15] and the references therein.

**Lemma 3.3** ([11]) Let $V = \{x \in X : \|x\| \leq \iota\} \ (\iota > 0)$, if $\{x(t), \ x \in V\}$ and $\{x'(t), \ x \in V\}$ are both equicontinuous on any compact intervals of $[0, +\infty)$ and equiconvergent at infinity, then $V$ is relatively compact on $X$. Where $V_1 \in \left\{\frac{x(t)}{1+t}, \ x \in V\right\} \cup \{x'(t), \ x \in V\}$ is equiconvergent at infinity if and only if for all $\epsilon > 0$, there exists $I = I(\epsilon)$ such that for all $x \in V_1$, it holds,
\[
\left|\frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2}\right| < \epsilon, \quad |x'(t_1) - x'(t_2)| < \epsilon,
\]
for all $t_1, t_2 \geq I$. 


Set $C = \phi q \int_{0}^{+\infty} h(s) ds$, \quad C(t_i) = \int_{0}^{t_i} \phi q \left( \int_{\tau}^{+\infty} h(s) ds \right) d\tau$, \quad \text{for } i = 1, 2.

**Lemma 3.4** Let $(H_1)$ and $(H_2)$ hold, Then $T : P \rightarrow P$ is completely continuous.

**Proof.** Firstly, we show that $T : P \rightarrow P$ is well defined. For any $x \in P$,

$$ (Tx)(t) \geq 0 \quad (3.3) $$

it can be easily seen that

$$ (Tx)''(t) = -h(s)f(t, x(t), x'(t)) \leq 0. \quad (3.4) $$

Combining (3.3) to (3.4), we can see that $T : P \rightarrow P$ is well defined.

Secondly, we aim to prove that $T$ is continuous and compact respectively.

Let $x_n \rightarrow x$ as $n \rightarrow +\infty$ in $P$, then there exists $r$ such that $\sup_{n \in \mathbb{N} \setminus \{0\}} \|x_n\| < r$. Set $B_r = \sup \{ f(t, (1 + t)u, v) \mid (t, u, v) \in [0, +\infty) \times [0, r]^2 \}$ and we have

$$ \int_{0}^{+\infty} h(s) |f(s, x_n(s), x_n'(s)) - f(s, x, x')| ds \leq 2B_r \int_{0}^{+\infty} h(s) ds. \quad (3.5) $$

Therefore, by the Lebesgue dominated convergence theorem, one arrives at

$$ |\phi_p(Tx_n)'(t) - \phi_p(Tx)'(t)| = \left| \int_{0}^{+\infty} h(s) (f(s, x_n(s), x_n'(s)) - f(s, x, x')) ds \right| $$

$$ \leq \int_{0}^{+\infty} h(s) |f(s, x_n(s), x_n'(s)) - f(s, x, x')| ds $$

$$ \rightarrow 0 \quad \text{as } n \rightarrow +\infty, $$

then, because $\phi_p$ is a continuous operator and it is monotone increasing, we have $(Tx_n)'(t) \rightarrow (Tx)'(t)$. Furthermore,

$$ \|(Tx_n)(t) - (Tx)(t)\| \leq M\|(Tx_n)'(t) - (Tx)'(t)\|_{\infty} \rightarrow 0 $$

as $n \rightarrow \infty$, $T$ is continuous.

$T$ is compact provided that it maps bounded sets into relatively compact sets. Let $\Omega$ be a bounded set of $P$, then there exists $\rho > 0$ such that $\|x\| < \rho$ for all $x \in \Omega$. Obviously, we have

$$ \|(Tx)'\|_{\infty} = \phi q \int_{0}^{+\infty} (h(s)f(s, x(s), x'(s)) ds \leq C\phi q(B_\rho) \quad (3.6) $$

for all $x \in \Omega$, where $B_\rho$ is defined as $B_r$. Then, $\|T\Omega\| \leq M\phi q(B_\rho)$. So $T\Omega$ is compact.
Moreover, for any $I \in (0, +\infty)$ and $t_1, t_2 \in [0, I)$,

$$
\left| \frac{(Tx)(t_1)}{1+t_1} - \frac{(Tx)(t_2)}{1+t_2} \right| \leq \int_0^{t_1} \left( \phi_q \int_\tau^{+\infty} h(s)f(s, x(s), x'(s))dsd\tau \right) \left| \frac{1}{1+t_1} - \frac{1}{1+t_2} \right| \\
+ \frac{1}{1+t_2} \int_0^{t_2} \left( \phi_q \int_\tau^{+\infty} h(s)f(s, x(s), x'(s))dsd\tau \right) \\
\leq \phi_q(B_{\rho}) \left( (I) \left| \frac{1}{1+t_1} - \frac{1}{1+t_2} \right| + (C(t_1) - C(t_2)) \right) \\
\to 0, \text{ uniformly as } t_1 \to t_2.
$$

(3.6)

for all $x \in \Omega$. So $T\Omega$ is equicontinuous on any compact interval of $[0, +\infty)$.

Finally, for any $x \in \Omega$,

$$
\lim_{t \to +\infty} \left| \frac{(Tx)(t)}{1+t} \right| = \lim_{t \to +\infty} \left( \frac{1}{1+t} \right) \int_0^{t} \left( \phi_q \int_\tau^{+\infty} h(s)f(s, x(s), x'(s))dsd\tau \right) \\
\leq M\phi_q(B_{\rho}) \lim_{t \to +\infty} \phi_q \left( \int_t^{+\infty} h(s)ds \right) = 0,
$$

(3.8)

$$
\lim_{t \to +\infty} |(Tx)'(t)| = \lim_{t \to +\infty} \phi_q \left( \int_t^{+\infty} h(s)f(s, x(s), x'(s))ds \right) \\
\leq \phi_q(B_{\rho}) \lim_{t \to +\infty} \phi_q \left( \int_t^{+\infty} h(s)ds \right) = 0.
$$

(3.9)

So $T\Omega$ is equiconvergent at infinity. $T\Omega$ is relatively compact, that is, $T$ is a compact operator.

Above all, $T: P \to P$ is completely continuous. Complete out proof.

4 Main Results

In this section, we impose growth conditions which allow us to apply Theorem 2.1 to the boundary value problem (1.1).

**Theorem 4.1** Suppose there exist positive numbers $a$, $b$, $d$ such that $0 < ka < b < \frac{2k-1}{2k(k+1)}d$ and $(H_1),(H_2)$ hold, further suppose that

$(H_3)$ $f(t, (1+t)u, v) \leq \phi_p(d/C)$, for $(t, u, v) \in [0, +\infty) \times [0, d] \times [0, d]$;

$(H_4)$ $f(t, (1+t)u, v) < \phi_p(a/C)$, for $(t, u, v) \in [0, +\infty) \times [0, a] \times [0, d]$;

$(H_5)$ $f(t, (1+t)u, v) > \phi_p(b/N_0)$, for $(t, u, v) \in \left[ \frac{1}{k}, k \right] \times \left[ \frac{k}{k}, \frac{2k(k+1)}{k-1}b \right] \times [0, d]$,
where \( N_0 = \frac{1}{(k+1)!} \int_0^k \phi_q \left( \int_{\tau}^{+\infty} h(s) ds \right) d\tau \). Then (1.1) has at least three positive solutions \( x_1, x_2, x_3 \) such that

\[
\sup_{0 \leq t < +\infty} x_i'(t) \leq d, \quad i = 1, 2, 3; \quad \sup_{0 \leq t < +\infty} \frac{x_3(t)}{1+t} < d \quad \text{with} \quad \min_{\frac{1}{k} \leq t \leq k} x_3(t) > \frac{k+1}{k} b.
\]

\[
\sup_{0 \leq t < +\infty} \frac{x_1(t)}{1+t} < a, \quad a < \sup_{0 \leq t < +\infty} \frac{x_2(t)}{1+t} < \frac{2(k+1)}{k-1} b \quad \text{with} \quad \min_{\frac{1}{k} \leq t \leq k} x_2(t) < \frac{k+1}{k} b,
\]

(4.1)

**Proof.** Let \( X, P, \alpha, \gamma, \theta, \psi, \) and \( T \) be defined as (2.1)-(2.4) respectively. Obviously, the fixed point of \( T \) coincide with the solutions of BVP (1.1) and (1.2). So it is enough to show that \( T \) has three positive fixed points on \( P \). Then we show that all the conditions of theorem 2.1 is satisfied.

Firstly, we show that the conditions \((S_1), (S_2)\) in Theorem 2.1 are satisfied. Set \( x(t) = (1 - e^{-t}) \frac{2k(k+1)}{2k-1} b, 0 \leq t < +\infty, \) and it is easy to check that \( x \in P(\alpha, b, \theta, c, \gamma, d) \) with \( \alpha(x) > b \), where \( c = \frac{2k(k+1)}{2k-1} b \). So \( \{ x \in P(\alpha, b, \theta, c, \gamma, d) \mid \alpha(x) > b \} \neq \emptyset \). Similarly, for \( x \in P(\alpha, b, \theta, c, \gamma, d) \), we have \( b \leq \frac{1}{1+t} \leq \frac{2(k+1)}{2k-1} b, t \in \left[ \frac{1}{k}, k \right] \) and \( 0 \leq x'(t) \leq d, t \in [0, +\infty) \). Therefore,

\[
\alpha(Tx) \geq \frac{1}{k+1} \theta(Tx) = \frac{1}{k+1} \sup_{0 \leq t < +\infty} (Tx)(t) = \frac{1}{k+1} \int_{\frac{1}{k}}^{k} \phi_q \left( \int_{\tau}^{k} h(s)f(s, x(s), x'(s)) ds \right) d\tau \\
\geq \frac{1}{(k+1)^2} \int_{\frac{1}{k}}^{k} \phi_q \left( \int_{\tau}^{k} h(s) ds \right) d\tau = \frac{b}{(k+1)^2 N_0} \int_{\frac{1}{k}}^{k} \phi_q \left( \int_{\tau}^{k} h(s) ds \right) d\tau = b.
\]

Hence, \( \alpha(Tx) > b \) for \( x \in P(\alpha, b, \theta, c, \gamma, d) \).

Secondly, we will prove that the condition \((S_2)\) of theorem 2.2 is fulfilled.

Let \( x \in P(\alpha, b, \gamma, d) \) with \( \theta(Tx) > \frac{2k(k+1)}{2k-1} b \), then we have

\[
\alpha(Tx) \geq \frac{1}{k+1} \theta(Tx) > \frac{1}{k+1} \frac{2k(k+1)}{2k-1} b > b.
\]

So both conditions \((S_1)\) and \((S_2)\) in Theorem 2.1 hold.

Finally, we show that condition \((S_3)\) of theorem 2.1 also holds. It can be seen clearly, \( \psi(0) = 0 < a \), there holds \( 0 \notin R(\psi_a, \gamma, d) \). Suppose that \( x \in R(\psi_a, \gamma, d) \) with \( \psi(x) = a \), Then, in view of assumption \((H_4)\) together with lemma 3.2 we have

\[
\psi(Tx) \leq \gamma(Tx) = (Tx)'(0) \leq \phi_q \left( \int_{0}^{+\infty} h(s)f(s, x(s), x'(s)) ds \right) \leq \frac{a}{C} \left( \int_{0}^{+\infty} h(s) ds \right) = a.
\]

So condition \((S_3)\) of theorem 2.1 is also satisfied. Therefore, an application of Theorem 2.1 implies the boundary value problem (1.1) has at least three
positive solutions $x_1, x_2, x_3$ such that $\psi(x_1) < a, \psi(x_2) > a$ with $\alpha(x_2) < b, \alpha(x_3) > b$.

In addition, condition (H2) guarantees those fixed points are positive. So BVP (1.1) (1.2) has at least three positive solutions $x_1, x_2, x_3$ satisfying (4.1) and we complete our proof.

With the same arguments as those in Theorem 3.1, we can complete our proof.

When $p = 2$, we can get the explicit existence results from the above theorem.

Set
\[
\overline{C} = \int_0^{+\infty} h(s)ds, \quad \overline{N} = \frac{1}{(k+1)^2} \left( \frac{\alpha \eta}{1 - \alpha} \int_1^k h(s)ds + \int_1^k \int_\tau^k h(s)dsd\tau \right),
\]
\[
\overline{N}_0 = \frac{1}{(k+1)^2} \int_1^k \int_\tau^k h(s)dsd\tau.
\]

Consider the following second order BVP on the half line
\[
\begin{aligned}
&x'' + h(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < +\infty, \\
&x(0) = \alpha x(\eta), \quad x'(\infty) = 0.
\end{aligned}
\]

**Corollary 4.2** Let $\alpha = 0$, suppose that there exist numbers $a, b,$ and $d$ such that $0 < ka < b < \frac{2k-1}{2k(k-1)}d$ and (H1), (H2), (H7), (H9) hold, Further suppose that

(H10) $f(t, (1 + t)u, v) > d/\overline{N}_0$ for \((t, u, v) \in \left[ \frac{b}{k}, k \right] \times \left[ \frac{b}{k}, \frac{2k(k+1)b}{k-1} \right] \times [0, d].$

Then (3.3) has at least three positive solutions $x_1, x_2, x_3$ such that (4.2) hold.

## 5 Example

**Example 5.1**

\[
\begin{aligned}
& (\phi_p(x'))' + e^{-4t}f(t, x(t), x'(t)) = 0, \quad 0 < t < +\infty, \\
& x(0) = 0, x'(\infty) = 0.
\end{aligned}
\]

where $p = 3, f(t, u, v) = \begin{cases} 
\frac{1}{20} |\sin t| + 10^5 \left( \frac{u}{1+t} \right)^{10} + \frac{1}{20} \left( \frac{u}{300} \right), & u \leq 1, \\
\frac{1}{20} |\sin t| + 10^5 \left( \frac{1}{1+t} \right)^{10} + \frac{1}{20} \left( \frac{u}{300} \right), & u \geq 1.
\end{cases}
\]

Set $h(t) = e^{-4t}$, and it can be easily seen that (H1) and (H2) hold. Let $k = 3, \ a = 1/3, \ b = 6, \ d = 300.$ Then after a series of calculation, we can
get that $C = 1/2$, $N > 1/32$, so the nonlinear term $f$ satisfies

$$f(t, (1 + t)u, v) \leq \frac{1}{20} + 10^5 + \frac{1}{20} < 3.6 \times 10^5 = \phi_3(d/C),$$

for $(t, u, v) \in [0, +\infty) \times [0, 600] \times [0, 300]$;

$$f(t, (1 + t)u, v) > 10^5 > \phi_3(b/N), \text{ for } (t, u, v) \in [\frac{1}{3}, 3] \times [2, 36] \times [0, 300];$$

$$f(t, (1 + t)u, v) < \frac{1}{9} = \phi_3(a/C), \text{ for } (t, u, v) \in [0, +\infty) \times [0, \frac{1}{3}] \times [0, 300].$$

Then the conditions in theorem 4.1 are all satisfied. So BVP (5.1) has at least three positive solutions $x_1$, $x_2$, $x_3$ such that

\[
\sup_{0 \leq t < +\infty} x_i(t) < 300, i = 1, 2, 3; \quad \sup_{0 \leq t < +\infty} \frac{x_1(t)}{1 + t} < \frac{1}{3}, \quad 2 < \sup_{0 \leq t < +\infty} \frac{x_2(t)}{1 + t} < 20, \\
\sup_{0 \leq t < +\infty} \frac{x_3(t)}{1 + t} < 300 \text{ with } \min_{\frac{1}{3} \leq t \leq 3} \frac{x_3(t)}{1 + t} > 8.
\]

References


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