Positive Solutions of Operator Equations on Half-Line

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Abstract

In this paper, under the weaker conditions, we investigate the problem of the existence of positive solutions for operator equations on half-line. We establish some results on the existence of multiple positive solutions for operator equations on half-line by applying the fixed-point theorem in a special function space. Our results significantly extend and improve many known results.

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The purpose of this paper is to establish the existence of positive solutions to the following operator equation on half-line:

\[ x(t) = Ax(t), \quad \forall \ t \in R^+. \] (1.1)

where A is an integral operator given by

\[ Ax(t) = \int_0^\infty G(t, s)m(s)f(s, x(s))ds, \quad \forall \ t \in R^+. \] (1.2)

and \( R^+ = [0, \infty) \), \( m(\cdot) \), \( f(\cdot) \) are given functions and the kernel \( G(t, s) \) of (1.2) is given by

\[ G(t, s) = \frac{1}{2k} \begin{cases} e^{-ks}(e^{kt} - e^{-kt}), & 0 \leq t \leq s, \\ e^{-kt}(e^{ks} - e^{-ks}), & 0 \leq s \leq t, \end{cases} \] (1.3)

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with \( k > 0 \) a constant.

In recent years, many authors([1-7]) are interested in the existence of positive solutions for some boundary value problems on half-line. In fact, in [1,3-7], some concrete equations of the special case where \( f \) is continuous at \( t=0 \) have extensively been studied. Only in [2], the nonlinear term is allowed to have singularities. To the author’s knowledge, there is little research concerning (1.1), so it is worthwhile to investigate operator equation (1.1).

The paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. Then, we give a result of completely continuous operator in Theorem 2.1. In Section 3, various conditions on the existence of multiple positive solutions to the operator equation (1.1) are discussed.

\( u \in C^2[0,1] \) is said to be a positive solution of operator equation (1.1) if and only if \( u \) satisfies operator equation (1.1) and \( u(t) > 0, \) for \( \forall \ t \in R^+. \)

2. Preliminaries and Lemmas

In this Section, we present some definitions and lemmas that will be used in the proof of our main results.

For convenience the readers, we present here the definitions of a cone and completely continuous operator.

**Definition 2.1.** A nonempty subset \( K \) of a Banach space \( E \) is called a cone if \( K \) is convex, closed, and

(i) \( \alpha x \in K \) for all \( \alpha \geq 0 \)

(ii) \( x, -x \in K \) implies \( x = \theta. \)

**Definition 2.2.** An operator \( F: E \to E \) is said to be completely continuous if \( F \) is continuous and maps bounded sets into precompact sets.

Obviously, we can see that the properties of the kernel given by (1.3) are as follows:

\[
G(t, s) \geq 0, \quad (t, s) \in R^+ \times R^+, \tag{2.1}
\]

\[
e^{-\mu t}G(t, s) \leq e^{-ks}G(s, s), \quad \mu \geq k(t, s) \in R^+ \times R^+, \tag{2.2}
\]

Let \( a \) and \( b \) be two numbers chosen at random from \((0, \infty)\). Without loss of generality, we may assume \( a < b \). We get

\[
G(t, s) \geq m_1 G(s, s)e^{-ks}, \quad (t, s) \in [a, b] \times R^+, \tag{2.3}
\]

where

\[
m_1 := \min \{e^{-kb}, e^{ka} - e^{-ka}\} < 1. \tag{2.4}
\]

In the paper, We shall consider the following space \( E \):

\[
E := \left\{ x \in C(R^+) : \sup_{t \in R^+} |x(t)|e^{-\lambda t} < \infty \right\}. \tag{2.5}
\]
where $\lambda > k$ is given. It is easy to testify that $E$ is a Banach space equipped with following Bielecki's norm
\[
\|x\| := \sup_{t \in \mathbb{R}^+} |x(t)e^{-\lambda t}|.
\] (2.6)
Meanwhile, we define a pone $P$ as follows:
\[
P := \left\{ x \in E : x(t) \geq 0, \ t \in (0, \infty); \min_{t \in [a,b]} x(t) \geq m_1\|x\| \right\}.
\] (2.7)

For convenience, let us list the following assumptions:

$(H_1)$ $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\sup_{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+} f(t,x) < \infty$.

$(H_2)$ $m : (0, \infty) \to \mathbb{R}^+$ is continuous and may be singular at $t = 0; m(t) \not\equiv 0$ on $\mathbb{R}^+$.

$(H_3)$
\[
0 < \int^b_a e^{-ks}G(s,s)m(s)ds, \int^\infty_0 e^{-ks}G(s,s)m(s)ds < \infty.
\]

$(H_4)$ $0 \leq f^0 < L, \ l_1 < f_\infty \leq \infty$.

$(H_5)$ $0 \leq f^\infty < L, \ l_1 < f_0 \leq \infty$.

$(H_6)$ $l_2 < f_0 \leq \infty$.

$(H_7)$ $l_2 < f_\infty \leq \infty$.

$(H_8)$ $0 \leq f^\infty < L$.

$(H_9)$ $0 \leq f^0 < L$.

$(H_{10})$ $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $f(t,x) \leq a(t) + b(t)x, (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$, where $a, b$ are continuous functions.

$(H_{11})$ The integrals $M_1 = \int^\infty_0 e^{-ks}a(s)m(s)ds$ and $M_2 = \int^\infty_0 e^{\lambda s}b(s)m(s)ds$ are convergent and $M_2 < 2K$.

In the above assumptions, we write
\[
f^\alpha = \lim_{\|x\| \to \alpha} \sup_{t \in \mathbb{R}^+} \frac{f(t,x)}{\|x\|}, \alpha = 0, \infty; \ f^{\beta}_\beta = \lim_{\|x\| \to \beta} \inf_{t \in \mathbb{R}^+} \frac{f(t,x)}{x}, \beta = 0, \infty,
\]
\[
L = \left( \int^\infty_0 e^{-ks}G(s,s)m(s)ds \right)^{-1},
\]
\[
l_1 = \left( m_1 \int^a_b e^{-ks}G(s,s)m(s)ds \right)^{-1}, l_2 = \frac{l_1e^{\lambda(a+b)/2}}{m_1}.
\]

**Lemma 2.1.** Let $P$ be a cone in a Banach space $X$ and $\Omega \subset X$ be a bounded set and $A : \overline{\Omega} \cap P \to P$ be a completely continuous operator. If $Ax \neq \lambda x$ for any $x \in \partial \Omega \cap P, \lambda \geq 1$, then the fixed point index $i(A, \Omega \cap P, P) = 1$. 
Lemma 2.2. Let $P$ be a cone in a Banach space $X$ and $\Omega \subset X$ be a bounded set and $A : \overline{\Omega} \cap P \to P$ be a completely continuous operator. If there exists a $u_0 > \theta$ such that
\[ x - Ax \neq tu_0x, \quad \forall x \in \partial \Omega \cap P, t \geq 0. \]
then we have $i(A, \Omega \cap P, P) = 0$.

Lemma 2.3 Let $X$ be a Banach space, and let $P$ be a cone in $X$. Assume $\Omega_1, \Omega_2$ are open subsets of $X$ with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$. Let $A : P \cap (\Omega_2 \setminus \Omega_1) \to P$ be a completely continuous operator, satisfying either

(i) $\|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial \Omega_1; \|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial \Omega_2.$

or

(ii) $\|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial \Omega_1; \|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial \Omega_2.$

Then $A$ have a fixed point in $P \cap (\Omega_2 \setminus \Omega_1)$.

Lemma 2.4 Let $E$ be the space given by
\[ E := \left\{ x \in C(R^+) : \sup_{t \in R^+} |x(t)|p(t) < \infty \right\}, \]
equipped with the norm
\[ \|x\| := \sup_{t \in R^+} \{ |x(t)|p(t) \}. \]
where $p; \quad \text{let } \Omega \subset E$. If the function $x \in \Omega$ are almost equicontinuous on $R^+$ (i.e., they are equicontinuous in each interval $[0, T], T \in (0, \infty)$ and uniformly bounded in the sense of the norm
\[ \|x\|_q := \sup_{t \in R^+} \{ |x(t)|q(t) \}. \]
where the function $q$ is positive and continuous on $R^+$ and
\[ \lim_{t \to \infty} \frac{p(t)}{q(t)} = 0. \]
then is $\Omega$ relatively compact in $E$.

In the following section, We will give a result of completely continuous operator.
Theorem 2.1 Assume that \((H_1) - (H_3)\) hold. Then for any bounded set \(\Omega \subset E\), we know that \(A : \overline{\Omega} \cap P \rightarrow P\) is completely continuous.

Proof. Let us choose any bounded set \(\Omega \subset E\).

Firstly, we prove that \(A : \overline{\Omega} \cap P \rightarrow P\). From \((H_2)\), we know that there exists \(t_0 \in (0, \infty)\) such that \(m(t_0) > 0\) or \(h(t_0) > 0\). Since \(m(t)\) or \(h(t)\) are continuous at \(t = t_0\), (1.2), (2.1) and \((H_1)\) imply that

\[
Ax(t) \geq 0, \ x \in \overline{\Omega} \cap P.
\] (3.1)

By \((H_1), (H_3)\) and (2.2), we get for any \(x \in \overline{\Omega} \cap P\) and \(t \in R^+\),

\[
|Ax(t)|e^{-\lambda t} = \int_{0}^{\infty} e^{-\lambda t} G(t, s)m(s)f(s, x(s))ds \leq \int_{0}^{\infty} e^{-ks}G(s, s)m(s)f(s, x(s))ds \leq \int_{0}^{\infty} e^{-ks}G(s, s)m(s)ds \sup_{(s, x) \in R^+} f(s, x).
\]

Inequalities (3.2) imply that

\[
\sup_{t \in R^+} \{ |Ax(t)|e^{-\lambda t} \} \leq \int_{0}^{\infty} e^{-ks}G(s, s)m(s)ds \sup_{(s, x) \in R^+} f(s, x) < \infty.
\]

Thus

\[
Ax \in E, \ \forall x \in \overline{\Omega} \cap P.
\] (3.3)

Moreover, for any \(x \in \overline{\Omega} \cap P\) and \(\sigma \in R^+\), we know by (2.2) and (2.3) that

\[
\min_{t \in [a, b]} Ax(t) = \min_{t \in [a, b]} \int_{0}^{\infty} G(t, s)m(s)f(s, x(s))ds \geq \int_{0}^{\infty} m_1e^{-ks}G(s, s)m(s)f(s, x(s))ds \geq \int_{0}^{\infty} m_1e^{-\lambda \sigma}G(s, s)m(s)f(s, x(s))ds = m_1e^{-\lambda \sigma}Ax(\sigma)
\]

Therefore,

\[
\min_{t \in [a, b]} Ax(t) \geq m_1\|Ax\|, \ x \in \overline{\Omega} \cap P.
\] (3.4)

(3.1), (3.3), and (3.4) tell us that \(A(\overline{\Omega} \cap P) \subset P\) as desired.

Now let us prove that \(A : \overline{\Omega} \cap P \rightarrow P\) is completely continuous when \(m : [0, \infty) \rightarrow [0, \infty), h : [0, \infty) \rightarrow [0, \infty]\) is continuous. For any \(\lambda_1\) such that \(k < \lambda_1 < \lambda\) and \(x \in \overline{\Omega} \cap P, t \in R^+\), we see similarly from the proof of (3.2) that functions \(\{Ax : x \in \overline{\Omega} \cap P\}\) are uniformly bounded with respect to the norm

\[
\|x\|_{\lambda_1} := \sup_{t \in R^+} \{ |x(t)|e^{-\lambda_1 t} \}.
\]

Moreover, for any \(T \in (0, \infty)\), the fact that

\[
G(t, s) \in C(R^+ \times R^+), \ m(t) \in C(R^+), \ f(t, x) \in C(R^+ \times R^+),
\]
and standard arguments tell us that \( \{ Ax : x \in \overline{\Omega} \cap P \} \) are equicontinuous in interval \([0, T]\). So \( \{ Ax : x \in \overline{\Omega} \cap P \} \) are almost equicontinuous on \( R^+ \). If we set \( p(t) = e^{-\lambda t} \), \( q(t) = e^{-\lambda_1 t} \), then Lemma 2.1 implies that \( A(\overline{\Omega} \cap P) \) is a precompact set in \( E \). Hence \( A \) is completely continuous.

In the end, we claim that operator \( A \) is also completely continuous if \( m, h : [0, \infty) \to [0, \infty) \) are singular at \( t=0 \). For each \( n \geq 1 \), denotes the operator \( A \) by

\[
A_n(x)(t) = \int_{1/n}^{\infty} G(t, s)m(s)f(s, x(s))ds, \quad x \in \overline{\Omega} \cap P, \quad t \in R^+. \quad (3.5)
\]

It follows from the above proof that

\[
A_n : \overline{\Omega} \cap P \to P \text{ is completely continuous, for each } n \geq 1. \quad (3.6)
\]

By \((H_1)\) and \((2.2)\), we have

\[
|A(x)(t) - A_n(x)(t)|e^{-\lambda t} = \int_0^{1/n} e^{-\lambda t}G(t, s)m(s)f(s, x(s))ds
\]

\[
\leq \int_0^{1/n} e^{-ks}G(s, s)m(s)f(s, x(s))ds
\]

\[
\leq \int_0^{1/n} e^{-ks}G(s, s)m(s)ds \cdot \sup_{(s, x) \in R^+ \times R^+} f(s, x).
\]

and so

\[
\sup_{t \in R^+} \{|A(x)(t) - A_n(x)(t)|e^{-\lambda t}\} \leq \int_0^{1/n} e^{-ks}G(s, s)m(s)ds \cdot \sup_{(s, x) \in R^+ \times R^+} f(s, x). \quad (3.7)
\]

Assumption \((H_3)\) and the absolute continuity of integral imply that

\[
\lim_{n \to \infty} \int_0^{1/n} G(s, s)m(s)ds = 0.
\]

Conditions \((3.6)-(3.8)\) justify that is \( A \) is also completely continuous. To sum up, the conclusion of Lemma 2.1 follows.

**Remark 2.1** Since \( m(t) \) is allowed to have singularity at \( t = 0 \) and \( t \) is in \([0, \infty)\), the proof of Theorem 2.1 has larger difference with those of the finite intervals.

3. Multiplicity results

In this section, we are concerned on the existence of at least two positive solutions of operator equation \((1.1)\). We obtain the following existence results.
**Theorem 3.1** Assume that \((H_1) - (H_4)\) hold, then operator equation (1.1) has at least one positive solution.

**Proof.** The conditions \(0 \leq f^0 < L\), imply that there exist \(s_1 > 0\) and \(\epsilon_1 > 0\) such that

\[
\sup_{t \in \mathbb{R}^+} f(t, x) \leq (L - \epsilon_1) \|x\|, 0 \leq \|x\| \leq s_1,
\]

Hence

\[
f(t, x) \leq (L - \epsilon_1) \|x\|, 0 \leq \|x\| \leq s_1, t \in \mathbb{R}^+,
\]  

(3.1)

Set \(\Omega_1 = \{x \in E : \|x\| < s_1\}\), then we get from (3.1) and \((H_3)\), for any \(x \in \partial\Omega_1 \cap P\)

\[
\|Ax\| = \sup_{t \in \mathbb{R}^+} \{\int_0^\infty G(t, s)m(s)f(s, x(s))ds|e^{-\lambda t}|\}
\leq (L - \epsilon_1)r_1 \sup_{t \in \mathbb{R}^+} \{\int_0^\infty e^{-\lambda t}G(t, s)m(s)ds\} \leq (L - \epsilon_1)r_1 \int_0^\infty e^{-ks}G(s, s)m(s)ds
\]
\[
= r_1L \int_0^\infty e^{-ks}G(s, s)m(s)ds - \epsilon_1 r_1 \int_0^\infty e^{-ks}G(s, s)m(s)ds < s_1
\]

Thus

\[
\|Ax\| < s_1, \forall x \in \partial\Omega_1 \cap P.
\]  

(3.2)

Under such circumstances, we may conclude that

\[
Ax \neq \lambda x, \forall x \in \partial\Omega_1 \cap P, \lambda \geq 1,
\]  

(3.3)

Otherwise, there would exist \(x_1 \in \partial\Omega_1 \cap P\) and \(\lambda_1 \geq 1\) such that \(Ax_1 = \lambda_1 x_1\). Thus

\[
\|Ax_1\| = \lambda_1 \|x_1\| \geq \|x_1\| = s_1.
\]  

(3.4)

Obviously, (3.4) is in contradiction with (3.2). This implies that (3.3) holds. Furthermore, assumptions \((H_1) - (H_3)\) tell us that Theorem 2.1 holds. Thus \(A : \overline{\Omega_1} \cap P \rightarrow P\) is completely continuous. Therefore, Lemma 2.1 and (3.3) mean

\[
i(A, \Omega_1 \cap P, P) = 1.
\]  

(3.5)

The conditions \(l_1 < f^{-} \leq \infty\) imply that there exist \(\eta_1 > m_1 s_1 > 0\) and \(\epsilon_2 > 0\) such that

\[
\inf_{t \in \mathbb{R}^+} f(t, x) \geq (l_1 + \epsilon_2)x, \|x\| \geq \eta_1,
\]

Hence

\[
f(t, x) \geq (l_1 + \epsilon_2)x, \|x\| \geq \eta, t \in [a, b].
\]  

(3.6)

Write

\[
s_2 = \frac{\eta_1}{m_1} > s_1, \Omega_2 = \{x \in E : \|x\| < s_2\}.
\]  

(3.7)
Let \( u_0 = 1 \in P \), then
\[
x - Ax \neq \lambda u_0, \quad \forall x \in \partial \Omega_2 \cap P, \forall \lambda \geq 0.
\] (3.8)

Suppose that (3.8) were false, then there would exist \( x_2 \in \partial \Omega_2 \cap P \) and \( \lambda_2 \geq 0 \) such that \( x_2 - Ax_2 = \lambda_2 \). Condition (3.6) and the fact that \( \|x_2\| = s_2 = \frac{n}{m_1} > \eta_1 \) imply that
\[
f(t, x_2(t)) \geq (l_1 + \epsilon_2)x_2(t), \quad t \in [a, b].
\] (3.9)

Set
\[
C_2 = \min \{x_2(t) : t \in [a, b]\}.
\] (3.10)

By virtue of (2.3), (3.9) and (3.10), we have for any \( t \in [a, b] \),
\[
x_2(t) = \int_0^\infty G(t, s)m(s)f(s, x_2(s))ds + \lambda_2 \geq \int_0^\infty G(t, s)m(s)f(s, x_2(s))ds
\]
\[
\geq \int_a^b G(t, s)m(s)f(s, x_2(s)))ds \geq \int_a^b m_1 e^{-ks}G(s, s)m(s)f(s, x_2(s))ds
\]
\[
\geq m_1 \int_a^b e^{-ks}G(s, s)m(s)(l_1 + \epsilon_2)x_2(s)ds
\]
\[
\geq m_1 (l_1 + \epsilon_2) \min_{t \in [a, b]} x_2(s) \int_a^b e^{-ks}G(s, s)m(s)ds
\]
\[
= C m_1 l_1 \int_a^b e^{-ks}G(s, s)m(s)ds + C \epsilon_2 m_1 \int_a^b e^{-ks}G(s, s)m(s)ds
\]
\[
= C + C \epsilon_2 m_1 \int_a^b e^{-ks}G(s, s)m(s)ds
\] (3.11)

(3.11) and \((H_3)\) imply that
\[
x_2(t) > C, \quad \forall t \in [a, b].
\] (3.12)

Obviously, (3.12) is in contradiction with (3.10). This implies that (3.8) holds. Therefore, Lemma 2.1 implies
\[
i(A, \Omega_2 \cap P, P) = 0.
\] (3.13)

Noting (3.5), (3.13) and the fact that \( \text{cl} \Omega_1 \subset \Omega_2 \), we have
\[
i(A, (\Omega_2 \setminus \text{cl} \Omega_1) \cap P, P) = i(A, \Omega_2 \cap P, P) - (A, \Omega_1 \cap P, P),
\]
\[
= 0 - 1 = -1,
\] (3.14)

(3.14) and the solution property of the fixed point index imply that the operator \( A \) has fixed points \( x \) which belongs to \((\Omega_2 \setminus \text{cl} \Omega_1) \cap P\) such that \( 0 < s_1 \leq \)
\[\|x_1\| \leq s_2,\text{it is clear that } x \text{ is a positive solution of operator equation (1.1)}.\]
The proof is completed. \(\Box\)

**Theorem 3.2** Assume that \((H_1)-(H_3)\) and \((H_5)\) hold, Then operator equation (1.1) has at least one positive solution.

**Proof.** The proof is similar to that of Theorem 3.1 and we omit it. \(\Box\)

**Theorem 3.3** Assume that \((H_1)-(H_3)\), \((H_6),(H_{10}),(H_{11})\) hold, Then operator equation (1.1) has at least two positive solutions.

**Proof.** Firstly, set \(r_1 = M_1(2k - M_2)^{-1},\) \(\Omega_{r_1} = \{x \in P : \|x\| < r_1\}.\)

Then we get from \(H_{10}\) and \((H_{11})\) that for any \(t \in R^+\) and \(x \in \partial \Omega_{r_1}\)

\[|Ax(t)|e^{-\lambda t} = \int_0^\infty e^{-\lambda t} G(t, s)m(s)f(s, x(s))ds \leq \int_0^\infty e^{-ks} G(s, s)m(s)f(s, x(s))ds\]
\[\leq \int_0^\infty e^{-ks} G(s, s)(a(s) + b(s)x(s))m(s)ds\]
\[\leq \int_0^\infty e^{-ks} G(s, s)a(s)m(s)ds + \int_0^\infty e^{-ks} G(s, s)b(s)x(s)m(s)ds\]
\[\leq \frac{1}{2k} \int_0^\infty e^{-ks} a(s)m(s)ds + \frac{1}{2k} \|x\| \int_0^\infty e^{(\lambda - k)s} b(s)m(s)ds\]
\[= \frac{1}{2k}(M_1 + r_1M_2) = r_1 = \|x\|\]

which indicates
\[\|Ax\| \leq \|x\|, \forall x \in \partial \Omega_{r_1}. \tag{3.24}\]

secondly, the condition \(l_2 < f_0 \leq \infty\) in \((H_6)\) tells us that there exists \(r_2 > r_1\) such that
\[f(t, x) \geq l_2x, t \in R^+, \|x\| \geq r_2. \tag{3.25}\]

Set \(\Omega_{r_2} = \{x \in P : \|x\| < r_2\}\).

Then we can see from (2.3)(2.7)and (3.25)that, for any \(x \in \partial \Omega_{r_2},\)

\[|Ax(\frac{a+b}{2})|e^{-\lambda(a+b)/2} = \int_0^\infty e^{-\lambda(a+b)/2} G(\frac{a+b}{2}, s)m(s)f(s, x(s))ds\]
\[\geq \int_a^b e^{-\lambda(a+b)/2} G(\frac{a+b}{2}, s)m(s)l_2x(s)ds\]
\[\geq e^{-\lambda(a+b)/2} \int_a^b G(\frac{a+b}{2}, s)m(s)l_2m_1\|x\|ds\]
\[\geq l_2m_1e^{-\lambda(a+b)/2} \int_a^b m_1e^{-ks} G(s, s)m(s)ds\|x\|
\[= \|x\|\]
Hence
\[ \|Ax\| \geq \|x\|, \forall x \in \partial \Omega_{r_2}. \] (3.27)

Finally, (3.24), (3.27) and Lemma 2.3 imply that the operator A has fixed points \( x \) which belongs to \( (\Omega_{r_2} \setminus \Omega_{r_1}) \cap P \) such that \( 0 < r_1 \leq \|x_3\| \leq r_2 \). It is clear that \( x \) is a positive solutions of operator equation (1.1). The proof is completed. \( \square \)

**Theorem 3.4** Assume that \((H_1) - (H_3)\) and \((H_7), (H_{10}), (H_{11})\) hold, Then operator equation (1.1) has at least two positive solutions.

**Proof.** The proof is similar to that of Theorem 3.3 and we omit it. \( \square \)

**Remark 3.1** In this paper, some results for positive solutions of operator equation (1.1) are obtained. Obviously, the conditions used in this paper is more extensive than the superlinear and sublinear conditions. Therefore, the paper generalizes and includes some known results.

**References**


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