On the Trivial and Nontrivial Cohomology with inner Symmetry groups of Some Classes of Operator Algebras

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Abstract. In this article we study the vanishing of the cohomology groups of some classes of $C^*$-algebra. We also give examples of nontrivial cohomology groups of operator algebra.

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1. 1-INTRODUCTION:

There are many studies interested in vanishing the cohomology group of operator algebras. For example, The third cohomology group $H^3(l^1(Z_+), l^1(Z_+)) = 0$ where $l^1(Z_+)$ is a unital semi-group algebra of $N$ [2], also the third cyclic cohomology group $HC^3(I, I) = 0$ where $I$ is a nonunital Banach algebra $l^1(Z)$ [2]. The dihedral cohomology $HD^n(A) = 0$, $n \in N$, $n$ is odd, $\epsilon = \pm 1$, where $A$ is biflat algebra [4]. The class of algebra called Amenable algebras, that, is all continuous derivation from an algebra $A$ into $A$-bimodule $M$ are inner, is a good result of the vanishing of the 1-st dimensional cohomology of a Banach algebra $A$, with coefficient in $A$-bimodule $M$ [7]. If $A$ is a $C^*$-algebra without bounded traces or a nuclear $C^*$-algebra, the Hochschild and dihedral cohomology groups vanish [1, 5].

There are examples of nontrivial cohomology groups: The even-dimensional simplicial, cyclic and dihedral cohomology of a nuclear $C^*$-algebra $A$ does not vanish [1, 10]. The dihedral cohomology of biflat algebra $A$ does not vanish but isomorphic to $HD^n(A) = \epsilon A^{tr}$, $n \in N$, $n$ is even, $\epsilon = \pm 1$ where $\epsilon A^{tr}$ is the set of all continuous traces on $A$ and $\epsilon A^{tr} = \epsilon A^{tr}[4]$. In the paper we study the vanishing cohomology groups (Reflexive and Dihedral) of some classes of $C^*$-algebra and give examples
of nontrivial dihedral cohomology groups of a commutative Banach algebra under special condition.

Following \[4\] the dihedral (co)homology is one of what called the cohomology with inner symmetry, that arise from the action of the dihedral group \(D_{n+1}\) of order \(2(n+1)\) on the Hochshild (simplicial) complex. We can also get the reflexive (co)homology by acting on the simplicial complex by the group \(Z/2\) of order 2 (The action of the cyclic group \(Z_{n+1}\) of order \((n+1)\) gives the cyclic (co)homology \([9]\)).

Firstly, we recall some definitions and facts from \([7, 8, 3]\).

Let \(A\) be a unital Banach algebra with an involution, \(M\) arbitrary \(A-\)bimodule. Denote by \(C^n(A, M)\) the Banach space of bounded (norm continuous) \(n\)-linear maps \(f\) from \(A\) into \(M\). These maps are called \(n\)-cochains. Consider the complex \(C(A) = (C^n(A, M), b^n)\) such that:

\[
C(A) : \to C^1(A) b_1 \to C^n(A) b^n C^{n+1}(A) b^{n+1} \to \ldots
\]

Where the operators \(b^n : C^n(A, M) \to C^{n+1}(A, M)\) is given by:

\[
b^n f(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) + (-1)^n f(a_n a_1, a_2, \ldots, a_{n-1})
\]

Clearly \(b^n = 0\) , so \(C(A)\) is a cochain complex and \(\Ker b^n \supseteq \text{Im } b^{n+1}\) that is \(B^n(A, M) \subseteq Z^n(A, M)\) \(B^n(A, M), Z^n(A, M)\) are the \(n-\)cocycles and \(n-\)coboundaries spaces. The group cohomology of algebra \(A\) with coefficient \(b-\)module \(M\) is givin by \(H^n(A, M) = \frac{Z^n(A, M)}{B^n(A, M)}\) , which is called the simplicial (Hochschild) cohomology if \(M = A^*\) and denoted by \(H^n(A, A^*) = HH(A)\), \(A^*\) \(A^*\) is dual space .

Note that \(Z^n(A, A^*)\) is always closed subspace of \(C^n(A, M)\), but the subspace \(B^n(A, A^*)\), in general , is not closed.

It is known that, \(C^n(A, A^*) \approx C^{n+1}(A, C)\) , where \(C^{n+1}(A, C)\) is the space of the bounded \((n+1)-\)linear forms in \(A\). The relation between \(C^n(A, A^*)\) and \(C^{n+1}(A, C)\) is given by:

\[
\langle a_0, \phi (a_1, \ldots, a_n) \rangle = \omega_\phi (a_0, a_1, \ldots, a_n), \phi \in C^n(A, A^*), \omega_\phi \in C^{n+1}(A, C),
\]

\[
\langle : A \times A^* \to C, \langle a, f \rangle = f(a).
\]

**Definition 1.** \((1.1).[8]\) Let \(A\) be Banach algebra, with an involution, over the field \(C\). The functional \(\omega \in C^{n+1}(A, C)\) is called \(\alpha\)-dihedral if it is satisfies the following conditions :

(i) Cyclic:

\[
\omega(a_0, a_1, \ldots, a_{n-1}, a_n) = (-1)^n \omega(a_n, a_0, \ldots, a_{n-1}),
\]
(ii) Reflexive:

\[
\omega(a_0, a_1, \ldots, a_n) = (-1)^{\frac{n(n+1)}{2}} \omega(a_0^*, a_1^*, \ldots, a_n^*)
\]

where \(a_i^*\) is the image of the elements \(a_i \in A, i = 0, 1, 2, \ldots, n\) under an involution \(*: A \to A\). The cochain \(f \in C^n(A)\) is called \(\alpha\)-dihedral if it coincides with its \(\alpha\)-dihedral \(\omega \in C^{n+1}(A, C)\). Suppose that \(CD^n(A)\) is the space of all dihedral \(n\)-cochains. Clearly, that the space \(CD^n(A)\) is invariant under the operator \(\delta^{n+1}: C^{n+1}(A, C) \to C^{n+2}(A)\).

**Definition 2.** (1.2). The \(n\)-dimensional cohomology of the space \(CD^n(A)\) is called the dihedral cohomology of an algebra \(A\) and shall be denoted by \(\alpha HD^n(A)\) where

\[
\alpha HD^n(A) = \frac{ZD^n(A)}{BD^n(A)}, \quad \alpha = \pm 1,
\]

\(ZD^n(A)\) and \(BD^n(A)\) are the dihedral \(n\)-cocycles \(n\)-coboundaries, which are given by the following:

\[
ZD^n(A) = CD^n(A) \cap Z^{n+1}(A, C), \quad BD^n(A) = CD^n(A) \cap B^{n+1}(A, C)
\]

The functional \(\omega \in C^{n+1}(A, C)\) is called \(\alpha\)-reflexive if it satisfies the condition (ii) in definition (1.2). We can similarly get the reflexive cohomology \(\alpha HR^n(A)\), \(\alpha = \pm 1\).

If we replace the Banach algebra by the \(C^*\)-algebra \(A\), we obtain the last cohomology groups of \(C^*\)-algebra.

2. **The Reflexive and Dihedral Cohomology of Stable \(C^*\)-Algebras.**

In this part we study the vanishing cohomology of some classes of \(C^*\)-algebra.

**Definition 3.** (2.1): The \(C^*\)-algebra \(A\) is stable if it isomorphic to tensor algebra \(K \times A\), where \(K\) is \(C^*\)-algebra of all compact operators on a countably infinite dimensional Hilbert space.

**Theorem 1.** (2.2): Let \(A\) be a stable \(C^*\)-algebra, then the reflexive and dihedral cohomology of \(A\) vanishes, i.e.

\[
\alpha HR^n(A) = 0, \quad \alpha HD^n(A) = 0, \quad \alpha = \pm 1.
\]

Firstly, we need the following facts:
Lemma 2. (2.3): [10] Let \( A \) be a \( C^* \)-algebra without unit, and for \( K \succ 0 \) let \( M_K(A) \) is the \( C^* \)-algebra of matrices over \( A \), and \( I : A \rightarrow M_K(A) \) is an inclusion map such that, \( a \rightarrow \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & & 0 \end{pmatrix} \), then \( I \) is a quasi-isomorphism, that is \( B^*(A) \Leftrightarrow B^*M_K(A) \).

Proof: Let \( A \) be a \( C^* \)-algebra without unit. If we adjoint \( A \) with an identity element we get \( \overline{A} = A \oplus C \).

Consider the following exact sequence

\[
0 \rightarrow A \rightarrow \overline{A} \rightarrow C \rightarrow 0
\]

where \( \overline{A} \) is algebra \( A \) with unit. We have the corresponding inclusion of algebra extensions

\[
0 \rightarrow A \oplus C \rightarrow M_K(A) \rightarrow M_K(C) \rightarrow 0
\]

Following [9], since \( M_K(A) \) is \( C^* \)-algebra, it is excision in Hochshild and cyclic homology. We can extend this fact to reflexive and dihedral cohomology. The diagram (10) induces the commutative diagram of short exact sequence:

\[
0 \rightarrow B^*(A) \rightarrow B^*(\overline{A}) \rightarrow B^*(C) \rightarrow 0
\]

\[
0 \rightarrow B^*(M_K(A)) \rightarrow B^*(M_K(\overline{A})) \rightarrow B^*(M_K(C)) \rightarrow 0
\]

since \( B^*(\overline{A}) \rightarrow B^*(M_K(\overline{A})) \) and \( B^*(C) \rightarrow B^*(M_K(C)) \) are isomorphisms in view of the Morita invariance in reflexive and dihedral cohomology, then \( B^*(A) \rightarrow B^*(M_K(\overline{A})) \).

Proposition 3. (2.4): Suppose that \( A \) is \( C^* \)-algebra, then the following isomorphism exists \( B^{n+1}(K \otimes A) \cong B(K \otimes q_i \otimes A) = 0 \), where \( q_i \), \( i = 0, 1, \ldots \) is the algebra of continuous functions on the \( n \)-sphere which vanishes at the North pole.

Proof: Consider the following exact sequence

\[
0 \rightarrow q_1 \rightarrow J_{p_i} C \rightarrow 0
\]
where $J$ is the algebra of continuous functions on the unit interval $[0,1]$, that vanishes at the left end, $Kerp = q_1$ and $p$ is a projection. Tensoring the sequence (10) by $(K \otimes A)$ we get the following exact and split sequence

(13) \[ 0 \to (K \otimes q_1 \otimes A) \to (K \otimes J \otimes A) \to (K \otimes A) \to 0 \]

The sequence (11) induces the long exact sequence in dihedral and reflexive cohomology:

(14) \[ \ldots \to B^{n+1}(K \otimes J \otimes A) \to B^{n+1}(K \otimes A) \partial \to B^n(K \otimes q_1 \otimes A) \to B^n(K \otimes J \otimes A) \to \ldots \]

where the connecting homomorphism $\partial$ commutes with the canonical maps: $^aHR^n, ^aHD^*$, $^aHR^n \to ^aHR^n$, and $^aHD^* \to ^aHD^*$, $\alpha = \pm 1$.

To show that $B^\ast(K \otimes J \otimes A) = 0$, consider for a C*-algebra $A$ a functor $F(A) = F(K \otimes A)$ from a category C*-algebra to a category of graded complex vector spaces. Clearly, $F$ is stable and splits-exact on the collection of the split C*-extensions [9]. It is known that any functor with these two properties (stable and split-exact) is homotopy invariant. Since the identity and zero endomorphisms of $(J \otimes A)$ are homotopic, then $(J \otimes A) = B^\ast(K \otimes J \otimes A) = 0$. From this result and sequence (14) we have $B^\ast(K \otimes A) = B^\ast(K \otimes J \otimes A) = 0$.

Proof theorem (2.2): From the last proposition we obtain the following commutative diagram,

(15) \[
\begin{array}{c}
\alpha HR^n(K \otimes A) \downarrow \rightarrow \alpha HR^n(K \otimes A) \\
\downarrow \rightarrow \alpha HD^0(K \otimes q_n \otimes A) \rightarrow \alpha HD^0(K \otimes q_n \otimes A)
\end{array}
\]

Thus, from the above diagram we obtain the isomorphism: $^aHR^n(K \otimes A) \cong ^aHD^n(K \otimes A)$. The Connes long exact sequence related the reflexive and dihedral cohomology is given by:

(16) \[ \ldots \to ^aHR^3(K \otimes A) \partial \to ^aHD^0(K \otimes A) \to ^aHD^2(K \otimes A) \to ^aHR^2(K \otimes A) \to ^aHD^1(K \otimes A) \to \ldots \to ^aHR^n(K \otimes A) \to ^aHD^{n-1}(K \otimes A) \partial \to ^aHD^n(K \otimes A) \to \ldots \]

where $s$ is a periodic operator. From the diagram (15) and (16) we have $^aHR^n(K \otimes A) \cong ^aHD^n(K \otimes A)$, $\alpha \neq 0$.

Example 1. (2.5): Let $U = \wp(H)/k$ be the Calkin algebra then, $^aHR^n(U) = ^aHD^n(U) = 0$. 

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Example 2. (2.6): Let $A$ be a $C^*$-algebra. A linear continuous functional $\tau: A \to C$ is called a bounded trace on algebra $A$, if $\tau$ satisfies the following conditions:

(i) $\tau(ab) = \tau(ba)$, $a, b \in A$

(ii) $\|\tau(ab)\| \leq \|a\|\|b\|

An example of algebra without bounded traces is the algebra of all bounded operators on Hilbert space.

Example 3. (2.7): Let $A$ be a $C^*$-algebra without bounded traces, then the dihedral cohomology group of $A$ vanishes. $\alpha HD^n(A) = 0$, $\alpha = \pm 1$.

Proof: see [1, 4].

3. 3- The dihedral cohomology of Banach algebras

In this part, we shall show that dihedral cohomology of a commutative Banach algebra $A$ with a nontrivial under the condition that $\text{codim} A^2 \geq n$, $n \geq 1$ does not vanish and show that the dihedral cohomology of nuclear $C^*$-algebra also is trivial.

Theorem 4. (3.1): Let $A$ be a commutative Banach algebra $A$ with a nontrivial involution under the condition $\text{codim} A^2 \geq n$, $n > 1$, the dihedral cohomology of $A$ does not vanish.

Lemma 5. (3.2): Let $A$ be a commutative Banach algebra, $X$ be a commutative Banach $A$-bimodule and $K \in B^n(A,X)$ then

\[
\sum (-1)\sigma K(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = 0
\]

(17)

Where $a_1, \ldots, a_n \in A$ $\sigma$, is a permutation of order $n$.

Proof: For a cochain $f \in C^{n+1}(A,X)$, that is $K = \delta^{n-1}f$, we show that

\[
\sum (-1)\sigma \delta^{n-1}f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = 0
\]

(18)

for any permutation and where

\[
\sigma(i+1) = \begin{cases} 
\tau(i) & i = 1, 2, \ldots, n-1 \\
\tau(n) & n = 0
\end{cases}
\]

(19)

the terms $a_{\sigma(1)}f(a_{\sigma(2)}, \ldots, a_{\sigma(n)})$ and $f(a_{\tau(1)}, \ldots, a_{\tau(n-1)})a_{\tau(n)}$ are equal and left hand side of relation (18) is the summation of pairs of terms which have the same values and the opposite signs. This fact is also true for the following:

\[
f(a_{\sigma(1)}, \ldots, a_{\sigma(K)}, a_{\sigma(K+1)}, \ldots, a_{\sigma(n)}), \quad f(a_{\tau(1)}, \ldots, a_{\tau(K)}, a_{\tau(K+1)}, \ldots, a_{\tau(n)})
\]

for every $K < n$ thus the summation (17) vanishes.
Proof of Theorem (3.1): since $n \leq \text{codim}A^2$, then there exists a linear independent elements $e_1, \ldots, e_n \in \frac{A}{A^2}$ define the functional $\varphi_i \in A^* = \text{Hom}_A(A, C)$, $1 \leq i \leq n + 1$ such that:

$$\varphi_i|A^2 = 0$$

$$\varphi_i(e_j) = \delta_{ij} = \begin{cases} 
1, & i \neq 1 \\
0, & i = j 
\end{cases}$$

Consider the cochain $f \in C^n(A, A^*)$ as follows,

$$f(a_1, \ldots, a_n) = j_1(a_1) \ldots j_n(a_n), \quad a_1, \ldots, a_n \in A.$$  

Clearly that $\delta f(a_1, \ldots, a_n) = 0$ hence $f \in Z^n(A, A^*)$ and the summation in (18) is equal to that is the cocycles does not equal to the coboundaries. Thus $H_n(A, A^*)$ does not vanish.

Now consider the co-chain $f \in C^n(A, A^*)$ such that:

$$f(a_1^*, \ldots, a_n^*) = j_2(a_1^*) \ldots j_3(a_2^*) \ldots j_{n+1}(a_n^*)j_1 + (-1)^n j_1(a_1^*) \ldots j_n(a_n^*)j_{n+1} +$$

$$\sum_{i=1}^n (-1)^n j_1(a_1^*) \ldots j_2(a_{i+1}^*) \ldots j_{n-i+1}(a_i^*)j_{n-i+3} + (a_1^*) \ldots j_{n-i+1}(a_{i-1}^*) \ldots j_{n-i+2}$$

A direct calculation shows that the functional $f \in Z^n(A)$ . Using lemma (3.2) the summation (18) does not vanish and hence $\alpha HD^n(A) \neq 0$, $\alpha = \pm 1$.

**Corollary 6.** (3.3). One of the reflexive cohomology $\alpha HR^n(A) \neq 0$, $\alpha = \pm 1$, under the conditions of theorem 2.4.

Proof: The same manner of theorem 2.4.

**Example 4.** (3.4). [5]

Let $A$ be a nuclear, then $\alpha HD^{2k}(A) \cong \alpha A^{tr}$ . Where $\alpha A^{tr}$ denotes the space of all bounded trace on $A$, $A^{tr} = \alpha A^{tr}$, $a \in A$, $\alpha = (-1)^k, k > 0$.

**References**


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