In this paper, we introduce \( rga \)-closed sets and \( rga \)-open sets and some of its basic properties.

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1 Introduction

N. Levine [14] introduced generalized closed sets in general topology as a generalization of closed sets. This concept was found to be useful and many results in general topology were improved. Many researchers like Balachandran, Sundaram and Maki [5], Bhattacharyya and Lahiri [6], Arockiarani [2], Dunham [11], Gnanambal [12], Malghan [18], Palaniappan and Rao [23], Park [24] Arya and Gupta [3] and Devi [8], Benchalli and wali [29] have worked on generalized closed sets, their generalizations and related concepts in general topology. In this paper, we define and study the properties of regular generalized \( \alpha \)-closed sets (briefly \( rga \)-closed). Moreover, in this paper we define \( rga \)-open sets and obtained some of its basic properties as results.

Throughout the paper \( X \) and \( Y \) denote the topological spaces \((X, \tau)\) and \((Y, \sigma)\) respectively and on which no separation axioms are assumed unless otherwise explicitly stated. For any subset \( A \) of a space \((X, \tau)\), the closure of \( A \), interior of \( A \), semi-interior of \( A \), semi-closure of \( A \), \( w \)-interior of \( A \), \( w \)-closure of \( A \), \( gpr \)-interior of \( A \), \( gpr \)-closure of \( A \), \( \alpha \)-closure of \( A \), \( \alpha \)-interior of \( A \) and the complement of \( A \) are denoted by \( cl(A) \) or \( \tau-cl(A) \), \( int(A) \) or
Definition 1.1. A subset $A$ of a space $X$ is called
1) a preopen set [20] if $A \subseteq \text{intcl}(A)$ and a preclosed set if $\text{clint}(A) \subseteq A$.
2) a semiopen set [13] if $A \subseteq \text{clint}(A)$ and a semiclosed set if $\text{intcl}(A) \subseteq A$.
3) a $\alpha$-open set [22] if $A \subseteq \text{intcl}(A)$ and a $\alpha$-closed set if $\text{clint}(A) \subseteq A$.
4) a semi-preopen set [1] if $A \subseteq \text{clint}(A)$ and a semi-preclosed set if $\text{intcl}(A) \subseteq A$.
5) a regular open set [28] if $A = \text{intcl}(A)$ and a regular closed set if $A = \text{clint}(A)$.

The intersection of all semiclosed (resp. semiopen) subsets of $X$ containing $A$ is called the semi-closure (resp. semi-kernel) of $A$ and is denoted by $\text{scl}(A)$ (resp. $\text{sker}(A)$). Also the intersection of all preclosed (resp. semi-preclosed and $\alpha$-closed) subsets of $X$ containing $A$ is called pre-closure (resp. semi-pre closure and $\alpha$-closure) of $A$ and is denoted by $\text{pcl}(A)$ (resp. $\text{spcl}(A)$ and $\alpha$-cl $(A)$).

Definition 1.2. A subset $A$ of a space $X$ is called
1) generalized closed set (briefly, $g$-closed) [14] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
2) semi-generalized closed set (briefly, $sg$-closed) [6] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi open in $X$.
3) generalized semiclosed set (briefly, $gs$-closed) [4] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
4) generalized $\alpha$-closed set (briefly, $g\alpha$-closed) [16] if $\text{acl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$.
5) $\alpha$-generalized closed set (briefly, $\alpha g$-closed) [15] if $\text{acl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
6) generalized semi-preclosed set (briefly, $gsp$-closed) [9] if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
7) regular generalized closed set (briefly, $rg$-closed) [23] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.
8) generalized preclosed set (briefly, $gp$-closed) [17] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
9) generalized preregular closed set (briefly, $gpr$-closed) [12] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.
10) weakly generalized closed set (briefly, $wg$-closed) [21] if $\text{clint}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$.
11) strongly generalized semi-closed set [25] (briefly, $g^*$-closed) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open in $X$.
12) $\pi$-generalized closed set (briefly, $\pi g$-closed) [10] if $\text{cl}(A) \subseteq U$ whenever $\tau$-int $(A)$, $\text{sint}(A)$, $\text{scl}(A)$, $\text{w-int}(A)$, $\text{w-cl}(A)$, $\text{gpr-int}(A)$, $\text{gpr-cl}(A)$, $\alpha$-int $(A)$, $\alpha$-cl $(A)$ and $A^C$ or $X - A$ respectively. $(X, \tau)$ will be replaced by $X$ if there is no chance of confusion.

Let us recall the following definitions as pre requesties.
A ⊆ U and U is π-open in X.
13) weakly closed set (briefly, w-closed)[27] if cl (A) ⊆ U whenever A ⊆ U and U is semiopen in X.
14) mildly generalized closed set (briefly, mildly g-closed)[24] if clint (A) ⊆ U whenever A ⊆ U and U is g-open in X.
15) semi weakly generalized closed set (briefly, swg-closed) if clint (A) ⊆ U whenever A ⊆ U and U is semiopen in X.
16) regular weakly generalized closed set (briefly, rwg-closed) if clint (A) ⊆ U whenever A ⊆ U and U is regular open in X.
17) A subset A of a space X is called regular semiopen[7] if there is a regular open U such U ⊂ A ⊂ cl(U). The family of all regular semiopen sets of X is denoted by RSO(X).

The complements of the above mentioned closed sets are their respective open sets.

2 rga-closed sets and their basic properties.

We introduce the following definition

Definition 2.1. A subset A of a space X is called regular α-open set (briefly, rα-open) if there is a regular open set U such that U ⊂ A ⊂ αcl(U).

The family of all regular α-open sets of X is denoted by RαO(X).

Definition 2.2. A subset A of a space X is called a regular generalized α-closed set (briefly, rga-closed) if αcl (A) ⊆ U whenever A ⊆ U and U is regular α-open in X. We denote the set of all rga-closed sets in X by RGA C(X).

First we prove that the class of rga-closed sets has properly lies between the class of gα-closed sets and the class of regular generalized closed sets.

Theorem 2.1. Every gα-closed set in X is rga-closed set in X, but not conversely.

Proof. The proof follows from the definitions and the fact that every regular open sets are regular α-open.

The converse of the above theorem need not be true, as seen from the following example.

Example 2.1. Let X = \{a, b, c, d, e\} with topology τ = \{X, \emptyset, \{a\}, \{d\}, \{e\}, \{a,d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}. Then the set A = \{a, d, e\} is rga-closed set but not gα-closed set in X.
**Theorem 2.2.** Every $w$-closed set in $X$ is $rg\alpha$-closed set in $X$, but not conversely.

**Proof.** The proof follows from the definitions and the fact that every regular $\alpha$-open set is semiopen and closed sets are $\alpha$-closed. □

The converse of the above theorem need not be true, as seen from the following example.

**Example 2.2.** Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Then the set $A = \{b\}$ is $rg\alpha$-closed set but not $w$-closed set in $X$.

**Theorem 2.3.** Every $rw$-closed set in $X$ is $rg\alpha$-closed set in $X$, but not conversely.

**Proof.** The proof follows from the definitions and the fact that closed sets are $\alpha$-closed. □

The converse of the above theorem need not be true, as seen from the following example.

**Example 2.3.** Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Then the set $A = \{b\}$ is $rg\alpha$-closed set but not $rw$-closed set in $X$.

**Theorem 2.4.** Every $rg\alpha$-closed set in $X$ is $rg$-closed set in $X$, but not conversely.

**Proof.** The proof follows from the definitions and the fact that every regular open sets are regular $\alpha$-open. □

The converse of the above theorem need not be true, as seen from the following example.

**Example 2.4.** Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Then the set $A = \{a, b\}$ is $rg$-closed set but not $rg\alpha$-closed set in $X$.

**Corollary 2.1.** Every closed set is $rg\alpha$-closed but not conversely.

**Proof.** Follows from sheik John [26] and theorem 2.3.

**Corollary 2.2.** Every regular closed set is $rg\alpha$-closed but not conversely.

**Proof.** Follows from stone [19] and corollary 2.1.
Corollary 2.3. Every rgα-closed set is a gpr-closed but not conversely.

Proof. Follows from Gnanmbal [12] and theorem 2.4.

Corollary 2.4. Every π-closed set is a rgα-closed set but not conversely.

Proof. Follows from [10] and corollary 2.1.

Theorem 2.5. Every rgα-closed set in X is rwg-closed set in X, but not conversely.

Proof. The proof follows from the definitions and the fact that every regular open sets are regular α-open.

The converse of the above theorem need not be true, as seen from the following example.

Example 2.5. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Then the set $A = \{a, b\}$ is rwg-closed set but not rgα-closed set in X.

Remark 2.1. From the above discussions and known results we have the following implications

In the following diagram, by

$A \rightarrow B$ we mean $A$ implies $B$ but not conversely and

$A \leftrightarrow B$ means $A$ and $B$ are independent of each other.
Theorem 2.6. The union of two rg$\alpha$-closed subsets of $X$ is also rg$\alpha$-closed subset of $X$.

Proof. Assume that $A$ and $B$ are rg$\alpha$-closed set in $X$. Let $U$ be regular $\alpha$-open in $X$ such that $A \cup B \subset U$. Then $A \subset U$ and $B \subset U$. Since $A$ and $B$ are rg$\alpha$-closed, $\alpha cl(A) \subset U$ and $\alpha cl(B) \subset U$. Hence $\alpha cl (A \cup B) = (\alpha cl(A)) \cup (\alpha cl(B)) \subset U$. That is $\alpha cl (A \cup B) \subset U$. Therefore $A \cup B$ is rg$\alpha$-closed set in $X$.

Remark 2.2. The intersection of two rg$\alpha$-closed sets in $X$ is generally not rg$\alpha$-closed set in $X$.

Example 2.6. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. If $A = \{a, b, c\}$ and $B = \{a, d, e\}$ Then $A$ and $B$ are rg$\alpha$-closed sets in $X$, but $A \cap B = \{a\}$ is not a rg$\alpha$-closed set in $X$. 
Theorem 2.7. If a subset $A$ of $X$ is $r\alpha$-closed set in $X$. Then $\alpha cl(A) \setminus A$ does not contain any nonempty regular $\alpha$-open set in $X$.

**proof.** Suppose that $A$ is $r\alpha$-closed set in $X$. We prove the result by contradiction. Let $U$ be a regular $\alpha$-open set such that $\alpha cl(A) \setminus A \supset U$ and $U \neq \phi$. Now $U \subset \alpha cl(A) \setminus A$. Therefore $U \subset X \setminus A$ which implies $A \subset X \setminus U$. Since $U$ is regular $\alpha$-open set, $X \setminus U$ is also regular $\alpha$-open in $X$. Since $A$ is $r\alpha$-closed set in $X$, by definition we have $\alpha cl(A) \subset X \setminus U$. So $U \subset X \setminus \alpha cl(A)$. Also $U \subset \alpha cl(A)$. Therefore $U \subset (\alpha cl(A) \cap (X \setminus \alpha cl(A))) = \phi$. This shows that, $U = \phi$ which is contradiction. Hence $\alpha cl(A) \setminus A$ does not contain any non-empty regular $\alpha$-open set in $X$.

The converse of the above theorem need not be true seen from following example.

**Example 2.7.** If $\alpha cl(A) \setminus A$ contains no non-empty $r\alpha$-open subset in $X$, then $A$ need not be $r\alpha$-closed set. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$ and $A = \{a, b\}$. Then $\alpha cl(A) \setminus A = \{a, b, c\} \setminus \{a, b\} = \{c\}$ does not contain non-empty regular $\alpha$-open set in $X$, but $A$ is not a $r\alpha$-closed set in $X$.

**Corollary 2.5.** If a subset $A$ of $X$ is $r\alpha$-closed set in $X$, then $\alpha cl(A) \setminus A$ does not contain any regular open set in $X$, but not conversely.

**Proof.** Follows from theorem 2.7. and the fact that every regular open set is regular $\alpha$-open.

**Corollary 2.6.** If a subset $A$ of $X$ is $r\alpha$-closed set in $X$, then $\alpha cl(A) \setminus A$ does not contain any non-empty regular closed set in $X$, but not conversely.

**Proof.** Follows from theorem 2.7. and the fact that every regular open set is regular $\alpha$-open.

**Theorem 2.8.** For an element $x \in X$, the set $X \setminus \{x\}$ is $r\alpha$-closed or regular $\alpha$-open.

**proof.** Suppose $X \setminus \{x\}$ is not regular $\alpha$-open set. Then $X$ is the only regular $\alpha$-open set containing $X \setminus \{x\}$. This implies $\alpha cl(X \setminus \{x\}) \subset X$. Hence $X \setminus \{x\}$ is $r\alpha$-closed set in $X$.

**Theorem 2.9.** If $A$ is regular open and $r\alpha$-closed then $A$ is regular closed and hence $\alpha$-clopen.

**proof.** Suppose $A$ is regular open and $r\alpha$-closed. As every regular open is regular $\alpha$-open and $A \subset A$, we have $\alpha cl(A) \subset A$. Also $A \subset \alpha clA$. Therefore $\alpha clA = A$. That is $A$ is $\alpha$-closed. Since $A$ is regular open, $A$ is $\alpha$-open. Now $cl(int(A)) = cl(A) = A$. Therefore $A$ is a regular closed and $\alpha$-clopen.
Theorem 2.10. If $A$ is $rg\alpha$-closed subset of $X$ such that $A \subseteq B \subseteq acl(A)$. Then $B$ is $rg\alpha$-closed set in $X$.

Proof. If $A$ is $rg\alpha$-closed subset of $X$ such that $A \subseteq B \subseteq acl(A)$. Let $U$ be a regular $\alpha$-open set of $X$ such that $B \subseteq U$. Then $A \subseteq U$. Since $A$ is a $rg\alpha$-closed we have $acl(A) \subseteq U$. Now $acl(B) \subseteq acl(acl(A)) = acl(A) \subseteq U$. Therefore $B$ is $rg\alpha$-closed set in $X$.

Remark 2.3. The converse of the theorem 2.10. need not be true in general. Consider the topological space $(X, \tau)$, where $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Let $A = \{b\}$ and $B = \{b, c\}$. Then $A$ and $B$ are $rg\alpha$-closed sets in $(X, \tau)$, but $A \subseteq B$ is not subset in $acl(A)$.

Theorem 2.11. Let $A$ be a $rg\alpha$-closed in $(X, \tau)$. Then $A$ is $\alpha$-closed if and only if $acl(A) \setminus A$ is a regular $\alpha$-open.

Proof. Suppose $A$ is a $\alpha$-closed in $X$. Then $acl(A) = A$ and so $acl(A) \setminus A = \phi$, which is regular $\alpha$-open in $X$. Conversely, suppose $acl(A) \setminus A$ is a regular $\alpha$-open set in $X$. Since $A$ is $rg\alpha$-closed, by theorem 2.7. $acl(A) \setminus A$ does not contain any nonempty regular $\alpha$-open in $X$. Then $acl(A) \setminus A = \phi$, hence $A$ is $\alpha$-closed set in $X$.

Theorem 2.12. If $A$ is regular open and $rg$-closed, then $A$ is $rg\alpha$-closed set in $X$.

Proof. Let $U$ be any regular $\alpha$-open set in $X$ such that $A \subseteq U$. Since $A$ is regular open and $rg$-closed, we have $acl(A) \subseteq A$. Then $acl(A) \subseteq A \subseteq U$. Hence $A$ is $rg\alpha$-closed set in $X$.

Theorem 2.13. If a subset $A$ of topological space $X$ is both regualr $\alpha$-open and $rg\alpha$-closed, then it is $\alpha$-closed.

Proof. Suppose a subset $A$ of topological space $X$ is both regualr $\alpha$-open and $rg\alpha$-closed. Now $A \subseteq A$. Then $acl(A) \subseteq A$. Hence $A$ is $\alpha$-closed.

Corollary 2.7. Let $A$ be regular $\alpha$-open and $rg\alpha$-closed subset in $X$. Suppose that $F$ is $\alpha$-closed set in $X$. Then $A \cap F$ is an $rg\alpha$-closed set in $X$.

Proof. Let $A$ be a regular $\alpha$-open and $rg\alpha$-closed subset in $X$ and $F$ be closed. By theorem 2.13., $A$ is $\alpha$-closed. So $A \cap F$ is a $\alpha$-closed and hence $A \cap F$ is $rg\alpha$-closed set in $X$.

Theorem 2.14. If $A$ is an open and $S$ is $\alpha$-open in topological space $X$, then $A \cap S$ is $\alpha$-open in $X$. 
Theorem 2.15. If $A$ is both open and $g$-closed set in $X$, then it is $rga$-closed set in $X$.

Proof. Let $A$ be an open and $g$-closed set in $X$. Let $A \subset U$ and let $U$ be a regular $\alpha$-open set in $X$. Now $A \subset A$. By hypothesis $\alpha cl(A) \subset A$. That is $\alpha cl(A) \subset U$. Thus $A$ is $rga$-closed in $X$. $\blacksquare$

Remark 2.4. If $A$ is both open and $rga$-closed in $X$, then $A$ need not be $g$-closed, in general, as seen from the following example.

Example 2.8. Consider $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. In this topological space the subset $\{a, d, e\}$ is an open and $rga$-closed set, but not $g$-closed.

Theorem 2.16. In a topological space $X$, if $R\alpha O(X) = \{X, \phi\}$, then every subset of $X$ is a $rga$-closed set.

Proof. Let $X$ be a topological space and $R\alpha O(X) = \{X, \phi\}$. Let $A$ be any subset of $X$. Suppose $A = \phi$. Then $\phi$ is $rga$-closed set in $X$. Suppose $A \neq \phi$. Then $X$ is the only regular $\alpha$-open set containing $A$ and so $\alpha cl(A) \subset X$. Hence $A$ is $rga$-closed set in $X$. $\blacksquare$

The converse of the theorem 2.16. need not be true in general as seen from the following example.

Example 2.9. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \phi, \{a, b\}, \{c, d\}\}$. Then every subset of $(X, \tau)$ is $rga$-closed set in $X$, But $R\alpha O(X, \tau) = \{X, \phi, \{a, b\}, \{c, d\}\}$.

Theorem 2.17. In a topological space $X$, $R\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$ iff every subset of $X$ is a $rga$-closed set.

Proof. Suppose that $R\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$. Let $A$ be any subset of $X$ such that $A \subset U$, where $U$ is a regular $\alpha$-open. Then $U \in R\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$. That is $U \in \{F \subset X : F^c \in \tau\}$. Thus $U$ is a $\alpha$-closed set. Then $\alpha cl(U) = U$. Also $\alpha cl(A) \subset \alpha cl(U) = U$. Hence $A$ is $rga$-closed set in $X$.

Conversely, suppose that every subset of $(X, \tau)$ is $rga$-closed. Let $U \in R\alpha O(X, \tau)$. Since $U \subset U$ and $U$ is $rga$-closed, we have $\alpha cl(U) \subset U$. Thus $\alpha cl(U) = U$ and $U \in \{F \subset X : F^c \in \tau\}$. Therefore $R\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$. $\blacksquare$

Definition 2.3. The intersection of all regular $\alpha$-open subsets of $(X, \tau)$ containing $A$ is called the regular $\alpha$-kernal of $A$ and is denoted by $rker(A)$.
Lemma 2.1. Let $X$ be a topological space and $A$ be a subset of $X$. If $A$ is a regular $\alpha$-open in $X$, then $r\alpha ker(A) = A$ but not conversely.

Proof. Follows from definition. 2.3.

Lemma 2.2. For any subset $A$ of $X$, $\alpha ker(A) \subset r\alpha ker(A)$.

Proof. Follows from the implication $R\alpha O(X) \subset \alpha O(X)$.

Lemma 2.3. For any subset $A$ of $X$, $A \subset r\alpha ker(A)$.

Proof. Follows from the definition. 2.3.

3 $rg\alpha$-open sets and $rg\alpha$-neighbourhoods.

In this section, we introduce and study $rg\alpha$-open sets in topological spaces and obtain some of their properties. Also, we introduce $rg\alpha$-neighbourhood (shortly $rg\alpha$-nbhd in topological spaces by using the notion of $rg\alpha$-open sets. We prove that every nbhd of $x$ in $X$ is $rg\alpha$-nbhd of $x$ but not conversely.

Definition 3.1. A subset $A$ in $X$ is called regular generalized $\alpha$-open (briefly, $rg\alpha$-open) in $X$ if $A^c$ is $rg\alpha$-closed in $X$. We denote the family of all $rg\alpha$-open sets in $X$ by $RG\alpha O(X)$.

Theorem 3.1. If a subset $A$ of a space $X$ is $w$-open then it is $rg\alpha$-open but not conversely.

Proof. Let $A$ be a $w$-open set in a space $X$. Then $A^c$ is $w$-closed set. By theorem 2.2. $A^c$ is $rg\alpha$-closed. Therefore $A$ is $rg\alpha$-open set in $X$.

The converse of the above theorem need not be true, as seen from the following example.

Example 3.1. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. In this topological space the subset $\{b\}$ is $rg\alpha$-open but not $w$-open.

Corollary 3.1. Every open set is $rg\alpha$-open set but not conversely.

Proof. Follows from sheik john [26] and theorem 3.1.

Corollary 3.2. Every regular open set is $rg\alpha$-open set but not conversely.

Proof. Follows from stone [28] and theorem 3.1.
Theorem 3.2. If a subset $A$ of a space $X$ is $rg\alpha$-open, then it is $rg$-open set in $X$.

Proof. Let $A$ be $rg\alpha$-open set in space $X$. Then $A^c$ is $rg\alpha$-closed set in $X$. By theorem 2.4., $A^c$ is $rg$-closed set in $X$. Therefore $A$ is $rg$-open set in space $X$. ■

The converse of the above theorem need not be true, as seen from the following example.

Example 3.2. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \varnothing, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. In this topological space the subset $\{a, b\}$ is $rg$-open but not $rg\alpha$-open set in $X$.

Theorem 3.3. If a subset $A$ of a space $X$ is $rg\alpha$-open, then it is $gpr$-open set in $X$, but not conversely.

Proof. Let $A$ be $rg\alpha$-open set in a space $X$. Then $A^c$ is $rg\alpha$-closed set in $X$. By corollary 2.3. $A^c$ is $gpr$-closed in $X$. Therefore $A$ is $gpr$-open set in $X$. ■

The converse of the above theorem need not be true, as seen from the following example.

Example 3.3. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \varnothing, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. In this topological space the subset $\{a, b\}$ is $gpr$-open but not $rg\alpha$-open.

Theorem 3.4. If a subset $A$ of a topological space $X$ is $rg\alpha$-open, then it is $rgw$-open set in $X$, but not conversely.

Proof. Let $A$ be $rg\alpha$-open set in a space $X$. Then $A^c$ is $rg\alpha$-closed set in $X$. By theorem 2.5. $A^c$ is $rgw$-closed in $X$. Therefore $A$ is $rgw$-open subset in $X$. ■

The converse of the above theorem need not be true, as seen from the following example.

Example 3.4. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \varnothing, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. In this topological space the subset $\{a, b\}$ is $rgw$-open set in $X$, but not $rg\alpha$-open.

Theorem 3.5. If $A$ and $B$ are $rg\alpha$-open sets in a space $X$. Then $A \cap B$ is also $rg\alpha$-open set in $X$.

Proof. If $A$ and $B$ are $rg\alpha$-open sets in a space $X$. Then $A^c$ and $B^c$ are $rg\alpha$-closed sets in a space $X$. By theorem 2.6. $A^c \cup B^c$ is also $rg\alpha$-closed set in $X$. That is $A^c \cup B^c = (A \cap B)^c$ is a $rg\alpha$-closed set in $X$. Therefore $A \cap B$ $rg\alpha$-open set in $X$. ■
Remark 3.1. The union of two $rg\alpha$-open sets in $X$ is generally not a $rg\alpha$-open set in $X$.

Example 3.5. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then $RG\alpha O(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $RaO(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Take $A = \{b, c, d\}$. Then $A$ is not $rg\alpha$-open. However $int (A) \cup A^c = \{b, c\} \cup \{a\} = \{a, b, c\}$. So for some regular $\alpha$-open $G$, we have $int (A) \cup A^c = \{a, b, c\} \subset G$ gives $G = X$, but $A$ is not $rg\alpha$-open.

Theorem 3.6. If a set $A$ is $rg\alpha$-open in a space $X$, then $G = X$, whenever $G$ is regular $\alpha$-open and $int (A) \cup A^c \subset G$.

Proof. Suppose that $A$ is $rg\alpha$-open in $X$. Let $G$ be regular $\alpha$-open and $int (A) \cup A^c \subset G$. This implies $G^c \subset (int (A) \cup A^c)^c = (int (A))^c \cap A$. That is $G^c \subset (int (A))^c - A^c$. Thus $G^c \subset cl (A)^c - A^c$. Since $(int (A))^c = cl (A^c)$. Now $G^c$ is also regular $\alpha$-open and $A^c$ is $rg\alpha$-closed, by theorem 2.7., it follows that $G^c = \phi$. Hence $G = X$.

The converse of the above theorem is not true in general as seen from the following example.

Example 3.6. Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then $RGAO(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $RaoO(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $RG\alpha O(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $RaO(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Take $A = \{b, c, d\}$. Then $A$ is not $rg\alpha$-open. However $int (A) \cup A^c = \{b, c\} \cup \{a\} = \{a, b, c\}$. So for some regular $\alpha$-open $G$, we have $int (A) \cup A^c = \{a, b, c\} \subset G$ gives $G = X$, but $A$ is not $rg\alpha$-open.

Theorem 3.7. Every singleton point set in a space is either $rg\alpha$-open or $r\alpha$-open.

Proof. Let $X$ be a topological space. Let $x \in X$. To prove $\{x\}$ is either $rg\alpha$-open or $r\alpha$-open. That is to prove $X - \{x\}$ is either $rg\alpha$-closed or $r\alpha$-open, which follows from theorem 2.8.

Analogous to a neighbourhood in space $X$, we define $rg\alpha$-neighbourhood in a space $X$ as follows.

Definition 3.2. Let $X$ be a topological space and let $x \in X$. A subset $N$ of $X$ is said to be a $rg\alpha$-nbhd of $x$ iff there exists a $rg\alpha$-open set $G$ such that $x \in G \subset N$.

Definition 3.3. A subset $N$ of space $X$, is called a $rg\alpha$-nbhd of $A \subset X$ iff there exists a $rg\alpha$-open set $G$ such that $A \subset G \subset N$. 
Remark 3.2. The \( r\alpha\)-nbhd \( N \) of \( x \in X \) need not be a \( r\alpha\)-open in \( X \).

Example 3.7. Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). Then \( R\alpha\theta O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\} \). Note that \( \{a, c\} \) is not a \( r\alpha\)-open set, but it is a \( r\alpha\)-nbhd of \( \{a\} \). Since \( \{a\} \) is a \( r\alpha\)-open set such that \( a \in \{a\} \subset \{a, c\} \).

Theorem 3.8. Every nbhd \( N \) of \( x \in X \) is a \( r\alpha\)-nbhd of \( X \).

Proof. Let \( N \) be a nbhd of point \( x \in X \). To prove that \( N \) is a \( r\alpha\)-nbhd of \( x \). By definition of nbhd, there exists an open set \( G \) such that \( x \in G \subset N \). As every open set is \( r\alpha\)-open set \( G \) such that \( x \in G \subset N \). Hence \( N \) is \( r\alpha\)-nbhd of \( x \).

Remark 3.3. In general, a \( r\alpha\)-nbhd \( N \) of \( x \in X \) need not be a nbhd of \( x \) in \( X \), as seen from the following example.

Example 3.8. Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). Then \( R\alpha\theta O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\} \). The set \( \{a, c\} \) is \( r\alpha\)-nbhd of the point \( c \), since the \( r\alpha\)-open sets \( \{c\} \) is such that \( c \in \{c\} \subset \{a, c\} \). However, the set \( \{a, c\} \) is not a nbhd of the point \( c \), since no open set \( G \) exists such that \( c \in G \subset \{a, c\} \).

Theorem 3.9. If a subset \( N \) of a space \( X \) is \( r\alpha\)-open, then \( N \) is a \( r\alpha\)-nbhd of each of its points.

Proof. Suppose \( N \) is \( r\alpha\)-open. Let \( x \in N \). We claim that \( N \) is \( r\alpha\)-nbhd of \( x \). For \( N \) is a \( r\alpha\)-open set such that \( x \in N \subset N \). Since \( x \) is an arbitrary point of \( N \), it follows that \( N \) is a \( r\alpha\)-nbhd of each of its points.

Remark 3.4. The converse of the above theorem is not true in general as seen from the following example.

Example 3.9. Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \). Then \( R\alpha\theta O(X) = \{X, \phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\} \). The set \( \{a, d\} \) is \( r\alpha\)-nbhd of the point \( a \), since the \( r\alpha\)-open set \( \{a\} \) is such that \( a \in \{a\} \subset \{a, d\} \). Also the set \( \{a, d\} \) is \( r\alpha\)-nbhd of the point \( \{d\} \). Since the \( r\alpha\)-open set \( \{d\} \) is such that \( d \in \{d\} \subset \{a, d\} \). That is \( \{a, d\} \) is \( r\alpha\)-nbhd of each of its points. However the set \( \{a, d\} \) is not a \( r\alpha\)-open set in \( X \).

Theorem 3.10. Let \( X \) be a topological space. If \( F \) is a \( r\alpha\)-closed subset of \( X \), and \( x \in F^c \). Prove that there exists a \( r\alpha\)-nbhd \( N \) of \( x \) such that \( N \cap F = \phi \).
Proof. Let $F$ be $rg\alpha$-closed subset of $X$ and $x \in F^c$. Then $F^c$ is $rg\alpha$-open set of $X$. So by theorem 3.9. $F^c$ contains a $rg\alpha$-nbhd of each of its points. Hence there exists a $rg\alpha$-nbhd $N$ of $x$ such that $N \subset F^c$. That is $N \cap F = \phi$.  

Definition 3.4. Let $x$ be a point in a space $X$. The set of all $rg\alpha$-nbhd of $x$ is called the $rg\alpha$-nbhd system at $x$, and is denoted by $rg\alpha-N(x)$.

Theorem 3.11. Let $X$ be a topological space and for each $x \in X$, Let $rg\alpha-N(x)$ be the collection of all $rg\alpha$-nbhds of $x$. Then we have the following results.

(i) $\forall x \in X$, $rg\alpha-N(x) \neq \phi$.
(ii) $N \in rg\alpha-N(x) \Rightarrow x \in N$.
(iii) $N \in rg\alpha-N(x)$, $M \supset N \Rightarrow M \in rg\alpha-N(x)$.
(iv) $N \in rg\alpha-N(x), M \in rg\alpha-N(x) \Rightarrow N \cap M \in rg\alpha-N(x)$.
(v) $N \in rg\alpha-N(x) \Rightarrow$ there exists $M \in rg\alpha-N(x)$ such that $M \subset N$ and $M \in rg\alpha-N(y)$ for every $y \in M$.

Proof. (i) Since $X$ is a $rg\alpha$-open set, it is a $rg\alpha$-nbhd of every $x \in X$. Hence there exists at least one $rg\alpha$-nbhd (namely $X$) for each $x \in X$. Hence $rg\alpha-N(x) \neq \phi$ for every $x \in X$.
(ii) If $N \in rg\alpha-N(x)$, then $N$ is a $rg\alpha$-nbhd of $x$. So by definition of $rg\alpha$-nbhd, $x \in N$.
(iii) Let $N \in rg\alpha-N(x)$ and $M \supset N$. Then there is a $rg\alpha$-open set $G$ such that $x \in G \subset N$. Since $N \subset M$, $x \in G \subset M$ and so $M$ is $rg\alpha$-nbhd of $x$. Hence $M \in rg\alpha-N(x)$.
(iv) Let $N \in rg\alpha-N(x)$ and $M \in rg\alpha-N(x)$. Then by definition of $rg\alpha$-nbhd there exists $rg\alpha$-open sets $G_1$ and $G_2$ such that $x \in G_1 \subset N$ and $x \in G_2 \subset M$. Hence $x \in G_1 \cap G_2 \subset N \cap M \rightarrow (1)$. Since $G_1 \cap G_2$ is a $rg\alpha$-open set, (being the intersection of two $rg\alpha$-open sets), it follows from (1) that $N \cap M$ is a $rg\alpha$-nbhd of $x$. Hence $N \cap M \in rg\alpha-N(x)$.
(v) If $N \in rg\alpha-N(x)$, then there exists a $rg\alpha$-open set $M$ such that $x \in M \subset N$. Since $M$ is a $rg\alpha$-open set, it is $rg\alpha$-nbhd of each of its points. Therefore $M \in rg\alpha-N(y)$ for every $y \in M$.

Theorem 3.12. Let $X$ be a nonempty set, and for each $x \in X$, let $rg\alpha-N(x)$ be a nonempty collection of subsets of $X$ satisfying following conditions.

(i) $N \in rg\alpha-N(x) \Rightarrow x \in N$
(ii) $N \in rg\alpha-N(x), M \in rg\alpha-N(x) \Rightarrow N \cap M \in rg\alpha-N(x)$.

Let $\tau$ consists of the empty set and all those non-empty subsets of $G$ of $X$ having the property that $x \in G$ implies that there exists an $N \in rg\alpha-N(x)$ such that $x \in N \subset G$, then $\tau$ is a topology for $X$.

Proof. (i) $\phi \in \tau$ by definition. We now show that $x \in \tau$. Let $x$ be any arbitrary element of $X$. Since $rg\alpha-N(x)$ is nonempty, there is an $N \in rg\alpha-N(x)$ and so $x \in N$ by (i). Since $N$ is a subset of $X$, we have $x \in N \subset X$.
Hence \( X \in \tau \).

(ii) Let \( G_1 \in \tau \) and \( G_2 \in \tau \). If \( x \in G_1 \cap G_2 \) then \( x \in G_1 \) and \( x \in G_2 \). Since \( G_1 \in \tau \) and \( G_2 \in \tau \), there exists \( N \in rga-N(x) \) and \( M \in rga-N(x) \), such that \( x \in N \subset G_1 \) and \( x \in M \subset G_2 \). Then \( x \in N \cap M \subset G_1 \cap G_2 \). But \( N \cap M \in rga-N(x) \) by (2). Hence \( G_1 \cap G_2 \in \tau \).

(iii) Let \( G_\lambda \in \tau \) for every \( \lambda \in \Lambda \). If \( x \in \bigcup \{ G_\lambda : \lambda \in \Lambda \} \), then \( x \in G_{\lambda x} \) for some \( \lambda x \in \Lambda \). Since \( G_{\lambda x} \in \tau \), there exists an \( N \in rga-N(x) \) such that \( x \in N \subset G_{\lambda x} \) and consequently \( x \in N \subset \bigcup \{ G_\lambda : \lambda \in \Lambda \} \). Hence \( \bigcup \{ G_\lambda : \lambda \in \Lambda \} \in \tau \). It follows that \( \tau \) is topology for \( X \).

**References**


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