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Modules with Seminorms which Take Values in a C^*-Algebra

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Abstract

In this paper the notation of topological Finsler module is introduced and properties of it is investigated. Let E be a left $\mathcal{A}$-module and $\mathcal{P}$ be a separating family of $\mathcal{A}$-valued Finsler seminorms on E. Then $\mathcal{P}$ is makes E to a topological Finsler module and every $\rho \in \mathcal{P}$ is continuous. In addition, if $\mathcal{A}$ is a commutative von Neumann algebra, (E, τ) is a topological Finsler $\mathcal{A}$-module such that E has a bounded open set and $\mathcal{P}$ induces τ. Then there exists a unique map $\rho : E \longrightarrow \mathcal{A}_+$ whose norm induces τ and makes E into a Finsler $\mathcal{A}$-module.

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1 Introduction

Finsler modules over C^*-algebras are generalization of Hilbert C^*- modules which is studied in [1] and [5] and we refer to [2] for the notion of the Hilbert C^*- modules. It is a useful tool in operator theory and operator algebras, and may be served as a noncommutative version of Banach bundles which are a main concept in Finsler geometry.

Definition 1.1. *Let $\mathcal{A}_+$ be the positive cone of a C^*-algebra $\mathcal{A}$ and E is a complex linear space which is a left $\mathcal{A}$-module (and $\lambda(ax) = (\lambda a)x = a(\lambda x)$ where $\lambda \in C, a \in \mathcal{A}$ and $x \in E$). An $\mathcal{A}$-valued Finsler seminorm is a map $\rho : E \to \mathcal{A}_+$ such that*

1) the map $\rho_E : x \to ||\rho(x)||$ is a seminorm on E, and

2) $\rho(ax)^2 = a\rho(x)^2a^*$ *for each* $a \in \mathcal{A}$ *and* $x \in E$.

An $\mathcal{A}$-valued Finsler seminorm is $\mathcal{A}$-valued Finsler norm if it satisfies $x = 0$ if $\rho(x) = 0$. E is equipped with a $\mathcal{A}$-valued Finsler norm is called a pre-Finsler $\mathcal{A}$-module. If $(E,\ \rho_E = ||.||_E)$ is complete then E is called a Finsler $\mathcal{A}$-module.

If we use the convention $|b| = (bb^*)^{1/2}$ for $b \in \mathcal{A}$, the condition (2) is equivalent to

$$\rho(ax) = |a\rho(x)|.$$

For $\mathcal{A}$ commutative this is the same as $\rho(ax) = |a|\rho(x)$, which is the usual form this sort of axiom takes in the commutative case. But this last version is not appropriate in the noncommutative case.

In this paper the notation of topological Finsler module is introduced and properties of it is investigated. Let E be a left $\mathcal{A}$-module and $\mathcal{P}$ be a separating family of $\mathcal{A}$-valued Finsler seminorms on E. Then $\mathcal{P}$ is makes E to a topological Finsler module and every $\rho \in \mathcal{P}$ is continuous. In addition, if $\mathcal{A}$ is a commutative von Neumann algebra, (E, τ) is a topological Finsler $\mathcal{A}$-module such that E has a bounded open set, and $\mathcal{P}$ induces τ. Then there exists a unique map $\rho : E \longrightarrow \mathcal{A}_+$ whose norm induces τ and makes E into a Finsler $\mathcal{A}$-module.

2 preliminaris

Definition 2.1. *Let* $\mathcal{A}_+$ *be the positive cone of a* C^**-algebra* $\mathcal{A}$ *and* E *is a complex linear space which is a left* $\mathcal{A}$*-module.* E *is said to be topological module whenever* E *is a topological vector space and module operation is continuous with respect to this topology.*

In view of [5 proposition 1] it follows that a Finsler module is a topological module. According to [6 Theorem 1.39] if E is a topological Finsler module, then E is normable if and only if E has a nonempty bounded open set.

Lemma 2.2. *Let* $\mathcal{A}$ *be a* C^**-algebra and* ρ *be an* $\mathcal{A}$*-valued Finsler seminorm on A-module* E *then* $\rho_E(ax) \leq ||a||\rho_E(x)$ *for all* $a \in \mathcal{A}$ *and* $x \in E$.

Proof. The condition (2) definition (1.1) implies that

$$\rho_E(ax) = ||\rho(ax)|| = |||a\rho(x)||| = ||a\rho(x)|| \leq ||a||||\rho(x)|| \leq ||a||\rho_E(x).\square$$

Theorem 2.3. *[8, Theorem 2.3]. Let* E *be a module over a* C^**-algebra* $\mathcal{A}$ *and* ρ *be a* $\mathcal{A}$*-valued Finsler seminorm on* E*. Then quotient* $\mathcal{A}$*-module* E/N_ρ

is a pre-Finsler $\mathcal{A}$-module with the module structure $a(x+N_\rho) = ax + N_\rho$ and $\rho^{'}(x+N_\rho) = \rho(x)$.

Theorem 2.4 *[8, Theorem 2.4]. Let $\mathcal{A}$ be a commutative C^*-algebra and ρ be an $\mathcal{A}$-valued Finsler seminorm on A-module E. Then ρ is subadditive and for all $x, y \in E$ we have,*

$$|\rho(x) - \rho(y)| \leq \rho(x-y).$$

3 main results

Definition 3.1. *A family $\mathcal{P}$ of $\mathcal{A}$-valued Fisler seminorms on E is said to be separating if to each $x \neq 0$ corresponds at least one $\rho \in \mathcal{P}$ with $\rho(x) \neq 0$.*

It is clear that $\mathcal{P}$ is separating if and only if $\mathcal{P}_E = \{\rho_E : \quad \rho \in \mathcal{P}\}$ is separating.

Theorem 3.2. *Let E be a left $\mathcal{A}$-module and $\mathcal{P}$ be a separating family of $\mathcal{A}$-valued Finsler seminorms on E. Then $\mathcal{P}$ is makes E to a topological Finsler module and every $\rho \in \mathcal{P}$ is continuous with respect to τ. Where τ is topology is induced of $\mathcal{P}_E$.*

Proof. In view of [6 Theorem 1.37] for the proof of the first section of the statement, is enough we prove that module operation is continuous with respect to τ. Suppose that $x_n \to x$ and $a_n \to a$. We must prove $a_n x_n \to ax$. For do this we show that $\forall \rho \in \mathcal{P}, \rho_E(a_n x_n - ax) \to 0$. By [6 Theorem 1.37] $\forall \rho \in \mathcal{P}, \quad \rho_E \in \mathcal{P}_E$ is continuous. Hence $\rho_E(x_n - x) \longrightarrow 0$ and by Lemma 2.2 we have

$$\rho_E(a_n x_n - ax) \leq ||a_n - a||\rho_E(x) + ||a||\rho_E(x_n - x) \to 0.$$

Now, we prove that every $\rho \in \mathcal{P}$ is continuous with respect to τ. Let ρ be fixed, let $x_n \longrightarrow x$. By Theorem 2.4 we have $|\rho_E(x_n) - \rho_E(x)| \longrightarrow 0$. let $c = sup\{\rho_E(x_n)\}$ then for any a in positive unit ball of $\mathcal{A}$ we conclude

$$\begin{aligned} |||a\rho(x_n)^2 a)|| - ||a\rho(x)^2 a||| &= |||\rho(ax_n)^2|| - ||\rho(ax)^2||| = |\rho_E(ax_n)^2 - \rho_E(ax)^2| \\ &= |\rho_E(ax_n) + \rho_E(ax)||\rho_E(ax_n) - \rho_E(ax)| \\ &\leq 2c||a|||\rho_E(x_n) - \rho_E(x)|. \end{aligned}$$

Now by [5, Theorem 4] this implies that

$$||\rho(x_n)^2 - \rho(x)^2|| \leq 2c|\rho_E(x_n) - \rho_E(x)| \longrightarrow 0.$$

So that we have $\rho(x_n)^2 \longrightarrow \rho(x)^2$.

Finally, by [4, Theorem 2.2.6], taking square roots implies that $\rho(x_n) \longrightarrow \rho(x)$. That is, ρ is continuous. □

Lemma 3.3. *Let E be an A-module over a C*-algebra $\mathcal{A}$ and $\rho, \rho^{'} : E \to A_+$ are two $\mathcal{A}$-valued Finsler seminorm which induce the same seminorm ρ_E. Then $\rho = \rho^{'}$.*

proof. Suppose $\rho(x) \neq \rho'(x)$ for some $x \in E$. Then $||\rho(x)^2 - \rho^{'}(x)^2|| \neq 0$ and so [5 Theorem 4] implies that there exists $a \in \mathcal{A}$ with $||a|| \leq 1$ such that $||a\rho(x)^2 a|| \neq ||a\rho^{'}(x)^2 a||$. Hence $\rho_E(ax)^2 = ||\rho(ax)^2|| \neq ||\rho^{'}(ax)^2|| = \rho^{'}_E(ax)^2$. This is a contradiction. □

Definition 3.4. *Let $\mathcal{A}$ be an unital C^*-algebra and E be a left $\mathcal{A}$-module.*
i) A set $C \subset E$ is said to be $\mathcal{A}$-convex if for all $x, y \in C$

$$ax + (I - a)y \in C \quad (0 \leq a \leq I).$$

ii) A set $B \subset E$ is said to be $\mathcal{A}$-balanced if for every $a \in \mathcal{A}$ which $|a| \leq I$, $aB \subset B$.

iii) A set $K \subset E$ is said to be $\mathcal{A}$-absorbing if for every $x \in E$ there exists $a > 0$ such that $x \in aK$.

It be shown that if $C \subset E$ is an $\mathcal{A}$-convex. Then C is a convex set. But it is not hold inverse the above results. Because $\mathcal{A}_+$ is a convex set but isn't $\mathcal{A}$-convex unless $\mathcal{A}$ is a commutative C^*-algebra.

It be shown that if B is $\mathcal{A}$-balanced then B is a balanced set. Because if $|\lambda| \leq 1$ then $\lambda B = \lambda IB \subset B$. But the inverse statement is not true. Since the set of all normal elements of $\mathcal{A}$ which is denoted by $\mathcal{A}_n$ is $\mathcal{A}$-balanced but it is not balanced. Because if $|\lambda| \leq 1$ then $\lambda\mathcal{A}_n \subset \mathcal{A}_n$. But if $0 < |a| \leq I$ then $a\mathcal{A}_n \subset \mathcal{A}_n$ is not necessary to be true.

The set $K \subset E$ is $\mathcal{A}$-absorbing if be absorbing. But the inverse statement is not true. Let $\mathcal{A}$ be a von Neumann algebra and K the set of all partial isometry in $\mathcal{A}$. If $x \in \mathcal{A}$ then there exists a partial isometry v and a positive operator k such that $x = vk$. That is, set of all partial isometry in $\mathcal{A}$ is $\mathcal{A}$-absorbing but isn't absorbing.

Theorem 3.5. *Let $\mathcal{A}$ be a commutative von Neumann algebra, let (E, τ) be a topological Finsler $\mathcal{A}$-module such that E has a bounded open set, let $\mathcal{P}$ be a separating family of $\mathcal{A}$-valued Finsler seminorms on E which induces τ. Then there exists a unique map $\rho : E \longrightarrow \mathcal{A}_+$ whose norm induces τ and makes E into a Finsler $\mathcal{A}$-module.*

proof. It is clear that 0 has a $\mathcal{A}$-convex $\mathcal{A}$-balanced and $\mathcal{A}$-absorbing neighborhood U. In view of [7 Theorem 3.3] we can consider minkoweski function of U with values in $\mathcal{A}$ which is a $\mathcal{A}$-valued Finsler seminorm on E. Also, $q_U(x) \neq 0$ if $x \neq 0$, that is, q_U is a $\mathcal{A}$-valued Finsler norm. Uniqueness of ρ follows from Lemma 3.3. □

Theorem 3.6. *Let (E, τ) be a topological Finsler module over a C^*-algebra $\mathcal{A}$, let $\mathcal{P}$ be a separating family of $\mathcal{A}$-valued Finsler seminorms on E which induce τ. Let I be an ideal of $\mathcal{A}$, let $\mathcal{B} = \mathcal{A}/I$, let $\pi : \mathcal{A} \to \mathcal{B}$ be the quotient map, and let $\rho' = \pi o \rho$. Then $IE = \cap ker\rho'$, E/IE is a $\mathcal{B}$-module, and $\{\rho'\}_{\rho\in\mathcal{P}}$ descends to a separating family of $\mathcal{B}$-valued Finsler seminorm on E/IE.*

proof. It is clear that E/IE is naturally a $\mathcal{B}$-module. If $a \in I$, $x \in E$ and $\rho \in \mathcal{P}$ then $\rho(ax)^2 = a\rho(x)^2a^* \in I$. Hence $\rho(ax) \in I$, that is, $ax \in ker\rho'$. this shows that $IE \subset \cap ker\rho'$. Conversely, if $x \in \cap ker\rho'$ then $\rho(x) \in I$ and so there exists a sequence $\{e_n\}$ of positive elements of I such that $e_n\rho(x) \to \rho(x)$. In the similarly to Lemma 12 of [5] we conclude that

$$\rho(x - e_nx)^2 = \rho(x)^2 - \rho(x)^2e_n - e_n\rho(x)^2 + e_n\rho(x)^2e_n.$$

It now follows that $\rho(x - e_nx)^2 \to 0$. Thus $\rho_E(x - e_nx) \to 0$ for every $\rho \in \mathcal{P}$ and so $x \in IE$. Therefore, it is shown that $IE = \cap ker(\rho')$.

By [5 Lemma 12] ρ' descends to a $\mathcal{B}$-valued Finsler seminorm on E/IE (see the proof). Since $IE = \cap ker(\rho')$, by Theorem 2.3 $\{\rho'\}_{\rho\in\mathcal{P}}$ descends to a separating family of $\mathcal{B}$-valued Finsler seminorm on E/IE which makes E/IE topological module. Consequently, E/IE is a topological Finsler $\mathcal{B}$-module. □

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