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A Note on (m, n) Quasi-Ideals in Semigroups

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Abstract. The notion of quasi-ideal is generalization of the notion of one sided ideal. It was introduced by Otto Steinfeld in 1953 for rings [6] and in 1956 for semigroups [7]. Quasi-ideals have been the theme of several papers ([4], [5], [2] etc). Furthermore, it has been widely studied in various algebraic structures viz. involution rings, near-rings, regular rings, Γ-Semirings, Lie algebra etc. In this paper, some intersection properties and characterizations of (m, n) quasi-ideals of semigroups has been taken into account. We have also obtained some results for left ideals, right ideals and quasi-ideals of semigroups and regular semigroups. Ideals in this paper have been taken into semigroup theoretical sense.

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1. Introduction and Prerequisites

A non-empty subset Q of a semigroup S is called a quasi-ideal of S if,

$$QS \cap SQ \subseteq Q\,.$$

Q is said to be an (m, n) quasi-ideal of S, if $S^m Q \cap QS^n \subseteq Q$. Further Q is said to be $(2, 3)$ quasi-ideal of S if $S^2 Q \cap QS^3 \subseteq Q$. Here the interesting fact is that a $(2, 3)$ quasi-ideal may not be a quasi-ideal. If we consider the class of quasi-ideals in semigroup we observe that it is the generalization of class of one-sided ideals in semigroups. So it is clear that every one sided ideal of a semigroup is a quasi-ideal of S.

Suppose that S is a semigroup then a left ideal, right ideal, two sided or a quasi-ideal Q of S is called proper if $B \neq 0$ and $B \neq S$ for any subsemigroup

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B of S. The intersection of a left ideal and a right ideal of a semigroup S is a quasi-ideal of S. However a quasi-ideal of a semigroup is not necessarily obtained in this way [5] and [4]. In this paper we'll show some of the results on (m, n) quasi-ideals of semigroups and regular semigroups. Also we will generalize some of the facts for left ideal, m-left ideal, right ideal, n-right ideal and quasi-ideals of semigroups.

EXAMPLE 1.1. ([9]) Let $SU_4(R)$ be the ring of all strictly upper triangular (4×4) matrices over R under the usual addition and multiplication of matrices and

$$Q = \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Bigg| \, x \in R$$

It is easy to prove that Q is a subring of $SU_4(R)$ (and hence subsemigroup in case of semigroup, as Q being a non-empty subset of S with $Q^2 \subseteq Q$)

We have that

$$SU_4(R)Q \cap QSU_4(R) = \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Bigg| \, x \in R \not\subseteq Q$$

but,

$$QSU_4(R)^2Q \cap QSU_4(R)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \subseteq Q \, .$$

This implies that Q is a $(2, 3)$-quasi-ideal but not a quasi-ideal of $SU_4(R)$.

If we replace ring by semigroup in Example 1.1, then it hold true.

LEMMA 1.2. *Suppose S be a semigroup and A_i be subsemigroup of S for all $i \in I$. Then $\cap_{i \in I} A_i$ is a subsemigroup of S.*

Proof. It is clear that $\cap_{i \in I} A_i$ is non-empty as $0 \in \cap_{i \in I} A_i$. Now to show that $\cap_{i \in I} A_i$ is subsemigroup of S we will show that closure and associative properties hold for $\cap_{i \in I} A_i$. Now suppose that $a, b \in \cap_{i \in I} A_i$. Since $a, b \in \cap_{i \in I} A_i$ it is obvious that $a, b \in A_i$ for all $i \in I$. Since A_i is a subsemigroup of S for all $i \in I$. Therefore $ab \in A_i$ for all $i \in I$. Thus $ab \in \cap_{i \in I} A_i$. In the similar way associative property also hold in A_i for any three elements of S and for all $i \in I$ And hence true for $\cap_{i \in I} A_i$. Thus $\cap_{i \in I} A_i$ is a subsemigroup of S.

PROPOSITION 1.3. *Suppose Q_i be an (m, n) quasi-ideal of S and A_i be a subsemigroup of S for all $i \in I$, then $A_i \cap Q_i$ is either empty or an (m, n) quasi-ideal of A_i.*

Proof. If $A_i \cap Q_i$ is not empty, then $A_i \cap Q_i$ is a subset of A_i such that $((A_i^m \cap Q_i)A_i^m) \cap (A_i^n(A_i^n \cap Q_i)) \subseteq A_i^2 \subseteq A_i$ and $((A_i^m \cap Q_i)A_i^m) \cap (A_i^n(A_i^n \cap Q_i)) \subseteq Q_iS^m \cap S^nQ_i \subseteq Q_i$. Which clearly shows that $A_i \cap Q_i$ is an (m,n) quasi-ideal of A_i for all $i \in I$.

PROPOSITION 1.4. *Suppose S be a semigroup and Q_i be an (m,n) quasi-ideals of S. Then the intersection of any set of (m,n) quasi-ideals of S is a quasi-ideal of S.*

Proof. Let Q_i be an (m,n) quasi-ideals of S then it is obvious that it contains the zero element of S. Therefore Q_i is not empty. Now let Q_i be any set of quasi-ideals of S for all $i \in I$. If $\cap_{i\in I}Q_i$ is not empty then for every Q_j for all $j \in I$, we have $E = (S^m(\cap_{i\in I}Q_i)) \cap ((\cap_{i\in I}Q_i)S^m) \subseteq S^nQ_j \cap Q_jS^n \subseteq Q_j$.

Hence $E \subseteq \cap_{i\in I}Q_i$ that shows that $\cap_{i\in I}Q_i$ is an (m,n) quasi-ideals of S.

THEOREM 1.5. *Suppose S is a semigroup and Q_i is an (m,n) quasi-ideal of S for all $i \in I$. Then $\cap_{i\in I}Q_i$ is an (m,n) quasi-ideal of S.*

Proof. By the above theorem it is clear that $\cap_{i\in I}Q_i$ is a subsemigroup of S. Consider, $d \in S^m(\cap_{i\in I}Q_i) \cap (\cap_{i\in I}Q_i)S^n$. Then we have $d = \Sigma x_kp_k = \Sigma y_lq_l$ for some $x_k \in S^m, y_l \in S^n$ also $p_k, q_l \in \cap_{i\in I}Q_i$. Thus for each k and l, we have that $p_k, q_l \in Q_i$ for all $i \in I$. Thus $d \in S^mQ_i \cap Q_iS^n$ for all $i \in I$. Since we know that Q_i is an (m,n) quasi-ideal of S for all $i \in I$ and $d \in \cap_{i\in I}Q_i$. And hence $\cap_{i\in I}Q_i$ is an (m,n) quais-ideal of S.

Suppose S is a semigroup and A is a subsemigroup of S then S is called an m-left ideal of S if $S^mA \subseteq A$ where m is any positive integer. Dually, if $AS^n \subseteq A$ then it is said to be n-right ideal of S where n is any positive integer. Now we'll show some of the results based on m-left and n-right quasi-ideal in semigroups.

THEOREM 1.6. *Let S be a semigroup then the following assertions are true:*

(i) *Suppose that A_i be an m-left ideal of S for all $i \in I$. Then $\cap_{i\in I}$ is an m-left ideal of S.*
(ii) *Suppose that B_i be an n-right ideal of S for all $i \in I$. Then $\cap_{i\in I}B_i$ is an n-right ideal of S.*

Proof. Since A_i is an m-left ideal of S for all $i \in I$. Therefore, $S^mA_i \subseteq A_i$. We will show that $\cap_{i\in I}A_i$ is also an m-left ideal of S. It is obvious that $0 \in \cap_{i\in I}A_i$. So, $\cap_{i\in I}A_i$ is non-empty. Next to show that $S^m(\cap_{i\in I}A_i) \subseteq \cap_{i\in I}A_i$. To show this we'll consider any $c \in S^m(\cap_{i\in I}A_i)$. Since $c \in S^m(\cap_{i\in I}A_i)$ therefore it is clear that $c \in S^m$ and $c \in \cap_{i\in I}A_i$. And hence $S^m(\cap_{i\in I}A_i) \subseteq \cap_{i\in I}A_i$. So $\cap_{i\in I}A_i$ is an m-left ideal of S. In the similar way one can show that $\cap_{i\in I}B_i$ is an n-right ideal of S.

THEOREM 1.7. *Suppose S be a semigroup and A, B be any two subsemigroup of S which are m-left and n-right ideals of S respectively. Then $A \cap B$ is also an (m,n) quasi-ideal of S.*

Proof. It is clear that $A \cap B$ is a subsemigroup of S. Now it remains to show only that $A \cap B$ is also an (m, n) quasi-ideal of S. i.e. to show that $(S^m(A \cap B)) \cap ((A \cap B)S^n) \subseteq S^m A \cap BS^n \subseteq A \cap B$, which proves that $A \cap B$ is an (m, n) quasi-ideals.

Q has (m, n) intersection property if Q is the intersection of an m-left ideal and an n-right ideal of S. In this case every m-left ideal and every n-right ideal have the (m, n) intersection property. Now we'll give a theorem which characterizes (m, n) quasi-ideals having the (m, n) intersection property.

THEOREM 1.8. *Suppose Q be an (m, n) quasi-ideal of a semigroup S. Then the following assertions are true:*

(i) *Q has the (m, n) intersection property.*
(ii) *$(Q \cup S^m Q) \cap (Q \cup QS^n) = Q$.*
(iii) *$S^m Q \cap (Q \cup QS^n) = Q$.*
(iv) *$QS^n \cap (Q \cup QS^n) \subseteq Q$.*

Proof. (i)$\Rightarrow$(ii). Let Q has the (m, n) intersection property. It is obvious that $Q \subseteq (Q \cup S^m Q) \cap (Q \cup QS^n)$. Now to show (ii) we'll show that $(Q \cup S^m Q) \cap (Q \cup QS^n) \subseteq Q$. As it is known that Q has the (m, n) intersection property. So there exist an m-left ideal A and n-right ideal B of S in such a way that $Q = A \cap B$. Thus $Q \subseteq A$ and $Q \subseteq B$. Also we have that $S^m Q \subseteq S^m A \subseteq A$ and in the similar way $QS^n \subseteq BS^n \subseteq B$ which implies that $Q \cup S^m Q \subseteq A$ and $Q \cup QS^n \subseteq B$. And hence we have that $(Q \cup S^m Q) \cap (Q \cup QS^n) \subseteq A \cap B = Q$. Thus (ii) is true.

Next we'll show that (ii)$\Rightarrow$(i). Consider, $(Q \cup S^m Q) \cap (Q \cup QS^n) = Q$. Then this is clear that both $Q \cup S^m Q$ and $Q \cup QS^n$ are m-left and n-right ideals of S as $S^m Q$ and QS^n both are m-left and n-right ideals of S. Thus by assumption Q has (m, n) intersection property. Hence in this way the proof completes.

(ii)$\Rightarrow$(iii) Consider $(Q \cup S^m Q) \cap (Q \cup QS^n) = Q$ then to show that $S^m Q \cap (Q \cup QS^n) \subseteq Q$. Since, $S^m Q \subseteq Q \cup S^n Q$, and also $S^m Q \cap (Q \cup QS^n) \subseteq (Q \cup S^m Q) \cap (Q \cup QS^n) = Q$. Next to show that

(iii)$\Rightarrow$(ii). Consider $S^m Q \cap (Q \cup QS^n) \subseteq Q$ then $Q \subseteq (Q \cup S^m Q) \cap (Q \cup QS^n)$. Thus we'll show that $(Q \cup S^m Q) \cap (Q \cup QS^n) \subseteq Q$. For this we suppose that $x \in (Q \cup S^m Q) \cap (Q \cup QS^n)$. Then we'll show that $x \in Q$ also. Now since $(Q \cup S^m Q) \cap (Q \cup QS^n)$ so suppose that $x = k + q_1 = l + q_2$ where $k \in S^m Q$, and $l \in QS^n$ and $q_1, q_2 \in Q$. So, $x = k + (q_2 - q_1) \in S^m Q \cap (Q \cup S^m Q)$. But by assumption we have $k \in Q$. And hence $x = k + q_1 \in Q$. So (ii) is proved.

The proofs for (ii)$\Rightarrow$(iv) and (iv)$\Rightarrow$ (ii) are almost similar to the proof of (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(ii), respectively.

PROPOSITION 1.9. *Suppose Q be an (m, n) quasi-ideal of S. If $S^m Q \subseteq QS^n$ or $QS^m \subseteq S^n Q$ then Q has the (m, n) intersection property.*

Proof. Consider $S^mQ \subseteq QS^n$ then $S^mQ = S^mQ \cap QS^n \subseteq Q$ which shows that Q is an m-left ideal of S. And so Q has the (m, n) intersection property. In the similar way if we consider $QS^m \subseteq S^nQ$ then Q has an n-right ideal of S. In this case also Q has the (m, n) intersection property. If arbitrary family of (m, n) quasi-ideal of S has the (m, n) intersection property then S is said to have intersection property of (m, n) quasi-ideal of S.

2. Regular Semigroups

Regular semigroups were introduced by J.A.Green [3]. Suppose S is a semigroup. Then an element $x \in S$ is said to be regular if there exist $y \in S$ such that $x = xyx$. The element $y \in S$ is said to be an inverse for x if $x = xyx$ and $y = yxy$. Semigroup S is said to be regular if all of its elements are regular. We also sometimes use the phrase Von Neumann regular semigroups for regular semigroups, as this phrase becomes very popular after the definition of Von Neumann regular rings (semigroups). In regular semigroup every principal ideal is generated by an idempotent element. It is known that every regular element of a regular semigroup has at least one inverse element.

EXAMPLE 2.1 ([1])**.** Let $S = M_{2\times 2}(K)$ be the set of the (2×2)-matrices over $K = Z/(2) = Z_2$ with the usual multiplication. Then S is regular and $Q = \left\{0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$ is a quasi-ideal of S. Also we obserb that Q is (m, n)-quasi-ideal of S.

Now we'll give intersection property of regular semigroup with (m, n) quasi-ideal.

THEOREM 2.2. *Every von Neumann regular semigroup has the intersection property of (m, n) quasi-ideals for any positive integer $m, n \in N$.*

Proof. Suppose that Q be an (m, n) quasi-ideal of a von Neumann regular semigroup S. Then it can be easily shown that $Q \subseteq QS^n$. Thus $Q \cup QS^n = QS^n$. Therefore, $S^mQ \cap (Q \cup QS^n) = S^mQ \cap QS^n \subseteq Q$. Thus by the Theorem 1.4, Q has the intersection property.

PROPOSITION 2.3 ([4], [8])**.** *A semigroup S is regular if and only if for any right ideal A and left ideal B of $S, AB = A \cap B$.*

The following proposition is the consequence of the above stated proposition.

PROPOSITION 2.4. *Suppose S be a von Neumann regular semigroup then a non-empty subset A of S is an (m, n) quasi-ideal of S if and only if it is the intersection of $(m, 0)$-right ideal and a $(0, n)$-left ideal of S.*

Proof. The proof is immediate by the definition of (m, n) ideal of S and hence $(m, 0)$ ideal and $(0, n)$ ideal of S. Let A be an (m, n) ideal of S i.e.

$A^mSA^n \subseteq A$. Then $(m,0)$ ideal of S is $A^mS \subseteq A$ and $(0,n)$ ideal is $SA^n \subseteq A$. Suppose that S be a von Neumann regular semigroup and A be a non-empty subset of S which is an (m,n) quasi-ideal of S i.e. $A^mS \cap SA^n \subseteq A$, which is possible only when A is the intersection of $(m,0)$ ideal and $(0,n)$ ideal of S which is obvious as $A^mS \cap SA^n \subseteq A$.

Now conversely suppose that $A^mS \cap SA^n \subseteq A$ then to show that S forms (m,n) ideal. As we know that A is an (m,n) ideal of S if $A^mSA^n \subseteq A$. Which is the product of m-left ideal and n-right ideal of S. Since $A^mS \subseteq A$ and $SA^n \subseteq A \Rightarrow A^mSSA^n = A^mSA^n \subseteq A$. Hence A is an (m,n) ideal of S.

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