Growth Estimates of Entire Functions
Based on Relative $L-(p,q)$th Order

Sanjib Kumar Datta
Department of Mathematics, University of North Bengal
Darjeeling, Pin-734013, West Bengal, India
sanjib_kr_datta@yahoo.co.in

Sanjib Mondal
Chaltia Sreeguru Pathksala High School
P.O.-Berhampore, Dist.-Murshidabad, PIN-742101
West Bengal, India
sanjib_mondal_math@yahoo.in

Abstract
In the paper we study the comparative growth properties of entire functions on the basis of relative $L-(p,q)$th order where $p,q$ are positive integers with $p > q$ and $L = L(r)$ is a slowly changing function.

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1 Introduction, Definitions and Notations.

Let $f$ and $g$ be two entire functions and $F(r) = \max \{|f(z)| : |z| = r\}$, $G(r) = \max \{|g(z)| : |z| = r\}$. If $f$ is non constant then $F(r)$ is strictly increasing and continuous and its inverse $F^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s \to \infty} F^{-1}(s) = \infty$.

Bernal [1] introduced the definition of relative order of $f$ with respect to $g$, denoted by $\rho_g(f)$ as follows:

$$\rho_g(f) = \inf \{\mu > 0 : F(r) < G(r^{\mu}) \ for \ all \ r > r_0(\mu) > 0\}$$

$$= \lim_{r \to \infty} \sup \frac{\log G^{-1}F(r)}{\log r}.$$
The definition coincides with the classical one [5] if \( g(z) = \exp z \). Similarly one can define the relative lower order of \( f \) with respect to \( g \) denoted by \( \lambda_g(f) \) as follows:

\[
\lambda_g(f) = \liminf_{r \to \infty} \frac{\log G^{-1} F(r)}{\log r}.
\]

Somasundaram and Thamizharasi [4] introduced the notions of \( L \)-order, \( L \)-lower order and \( L \)-type for entire functions where \( L = L(r) \) is a positive continuous function increasing slowly i.e., \( L(ar) \sim L(r) \) as \( r \to \infty \) for every positive constant ‘a’. Their definitions are as follows:

**Definition 1.** [4] The \( L \)-order \( \rho_L^f \) and the \( L \)-lower order \( \lambda_L^f \) of an entire function \( f \) are defined as follows:

\[
\rho_L^f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log [rL(r)]} \quad \text{and} \quad \lambda_L^f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log [rL(r)]},
\]

where \( \log^{[k]} x = \log(\log^{[k-1]} x) \) for \( k = 1, 2, 3, \ldots \) and \( \log^{[0]} x = x \).

When \( f \) is meromorphic, then

\[
\rho_L^f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [rL(r)]} \quad \text{and} \quad \lambda_L^f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log [rL(r)]}.
\]

**Definition 2.** [4] The \( L \)-type \( \sigma_L^f \) of an entire function \( f \) with \( L \)-order \( \rho_L^f \) is defined as

\[
\sigma_L^f = \limsup_{r \to \infty} \frac{\log M(r, f)}{[rL(r)]^{\rho_L^f}}, \quad 0 < \rho_L^f < \infty.
\]

For meromorphic \( f \), the \( L \)-type \( \sigma_L^f \) becomes

\[
\sigma_L^f = \limsup_{r \to \infty} \frac{T(r, f)}{[rL(r)]^{\rho_L^f}}, \quad 0 < \rho_L^f < \infty.
\]

Juneja, Kapoor and Bajpai [3] defined the \((p,q)\)th order and the \((p,q)\)th lower order of an entire function \( f \) respectively as follows:

\[
\rho_f(p, q) = \limsup_{r \to \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} r},
\]

and

\[
\lambda_f(p, q) = \liminf_{r \to \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} r}.
\]
When \( f \) is meromorphic, one can easily verify that
\[
\rho_f(p, q) = \limsup_{r \to \infty} \frac{\log[p] T(r, f)}{\log[q] r}
\]
and
\[
\lambda_f(p, q) = \liminf_{r \to \infty} \frac{\log[p] T(r, f)}{\log[q] r},
\]
where \( p, q \) are positive integers and \( p > q \).

So with the help of the above notion one can easily define the relative \( L - (p, q) \)th order and relative \( L - (p, q) \)th lower order of entire functions.

**Definition 3.** The relative \( L - (p, q) \)th order and relative \( L - (p, q) \)th lower order of an entire function \( f \) with respect to another entire function \( g \) respectively denoted by \( L\rho_g^f(p, q) \) and \( L\lambda_g^f(p, q) \) are defined as
\[
L\rho_g^f(p, q) = \limsup_{r \to \infty} \frac{\log[p] G^{-1} F(r)}{\log[q] [rL(r)]}
\]
and
\[
L\lambda_g^f(p, q) = \liminf_{r \to \infty} \frac{\log[p] G^{-1} F(r)}{\log[q] [rL(r)]},
\]
where \( p, q \) are positive integers and \( p > q \).

The more generalised concept of \( L - \) order and \( L - \) type of entire and meromorphic functions are \( L^* - \) order and \( L^* - \) type respectively. Their definitions are as follows:

**Definition 4.** The \( L^* - \) order, \( L^* - \) lower order and \( L^* - \) type of a meromorphic function \( f \) are defined by
\[
\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [re^{L(r)}]}, \quad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log [re^{L(r)}]}
\]
and
\[
\sigma_f^{L^*} = \limsup_{r \to \infty} \frac{\log \{2\} M(r, f)}{\log \{2\} [re^{L(r)}]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.
\]

When \( f \) is entire, one can easily verify that
\[
\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log[2] M(r, f)}{\log [re^{L(r)}]}, \quad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log[2] M(r, f)}{\log [re^{L(r)}]}
\]
and
\[
\sigma_f^{L^*} = \limsup_{r \to \infty} \frac{\log M(r, f)}{\log \{2\} [re^{L(r)}]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.
\]
Definition 5. The relative $L^* - (p,q)$th order $L^* \rho_g^f(p,q)$ and the relative $L^* - (p,q)$th lower order $L^* \lambda_g^f(p,q)$ of an entire function $f$ with respect to another entire function $g$ are defined as

$$L^* \rho_g^f(p,q) = \limsup_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[q]} [re^{L(r)}]}$$

and

$$L^* \lambda_g^f(p,q) = \liminf_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[q]} [re^{L(r)}]},$$

where $p, q$ are positive integers and $p > q$.

In the paper we establish some results on the growth properties of entire functions on the basis of relative $L^* - (p,q)$th order and relative $L^* - (p,q)$th lower order where $p, q$ are positive integers with $p > q$. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions as those are available in [6] and [2].

2 Theorems.

In this section we present the main results of the paper.

In the following theorems we see the application of relative $L^* - (p,q)$th order and relative $L^* - (p,q)$th lower order in estimating the growth properties of entire functions where $p, q$ are positive integers with $p > q$.

Theorem 1. Let $f, g$ and $h$ be three entire functions such that $0 < L^* \lambda_g^f(p,q) \leq L^* \rho_g^f(p,q) < \infty$ and $0 < L^* \lambda_g^h(m,q) \leq L^* \rho_g^h(m,q) < \infty$. Then

$$\frac{L^* \lambda_g^f(p,q)}{L^* \rho_g^h(m,q)} \leq \liminf_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)} \leq \frac{L^* \lambda_g^f(p,q)}{L^* \lambda_g^h(m,q)},$$

$$\leq \limsup_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)} \leq \frac{L^* \rho_g^f(p,q)}{L^* \lambda_g^h(m,q)},$$

where $p, q, m$ are positive integers with $q < \min \{p, m\}$.

Proof. From the definition of relative $L^* - (p,q)$th order and relative $L^* - (p,q)$th lower order we have for arbitrary positive $\epsilon$ and for all large values of $r$,

$$\log^{[p]} G^{-1} F(r) \geq (L^* \lambda_g^f(p,q) - \epsilon) \log^{[q]} [rL(r)] \quad (1)$$

and

$$\log^{[m]} G^{-1} H(r) \leq (L^* \rho_g^h(m,q) + \epsilon) \log^{[q]} [rL(r)]. \quad (2)$$
Now from (1) and (2) it follows for all large values of \(r\),
\[
\frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)} \geq \frac{L \lambda^f_g(p, q) - \epsilon}{L \rho^h_g(m, q) + \epsilon}.
\]
As \(\epsilon (> 0)\) is arbitrary, we obtain that
\[
\liminf_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)} \geq \frac{L \lambda^f_g(p, q)}{L \rho^h_g(m, q)}.
\]
(3)

Again for a sequence of values of \(r\) tending to infinity,
\[
\log^{[p]} G^{-1} F(r) \leq (L \lambda^f_g(p, q) + \epsilon) \log^{[q]} [rL(r)]
\]
(4)

and for all large values of \(r\),
\[
\log^{[m]} G^{-1} H(r) \geq (L \lambda^h_g(m, q) - \epsilon) \log^{[q]} [rL(r)].
\]
(5)

So combining (4) and (5) we get for a sequence of values of \(r\) tending to infinity,
\[
\frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)} \leq \frac{L \lambda^f_g(p, q) + \epsilon}{L \lambda^h_g(m, q) - \epsilon}.
\]

Since \(\epsilon (> 0)\) is arbitrary it follows that
\[
\liminf_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)} \leq \frac{L \lambda^f_g(p, q)}{L \lambda^h_g(m, q)}.
\]
(6)

Also for a sequence of values of \(r\) tending to infinity,
\[
\log^{[m]} G^{-1} H(r) \leq (L \lambda^h_g(m, q) + \epsilon) \log^{[q]} [rL(r)].
\]
(7)

Now from (1) and (7) we obtain for a sequence of values of \(r\) tending to infinity,
\[
\frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)} \geq \frac{L \lambda^f_g(p, q) - \epsilon}{L \lambda^h_g(m, q) + \epsilon}.
\]
Choosing \(\epsilon \to 0\) we get that
\[
\limsup_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[q]} G^{-1} H(r)} \geq \frac{L \lambda^f_g(p, q)}{L \lambda^h_g(m, q)}.
\]
(8)

Also for all large values of \(r\),
\[
\log^{[p]} G^{-1} F(r) \leq (L \rho^f_g(p, q) + \epsilon) \log^{[q]} [rL(r)].
\]
(9)
So from (5) and (9) it follows for all large values of $r$, 

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{L \rho_f^h(p, q)}{L \lambda_g^h(m, q)} + \epsilon.$$ 

As $\epsilon (> 0)$ is arbitrary we obtain that

$$\limsup_{r \to \infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{L \rho_f^h(p, q)}{L \lambda_g^h(m, q)}.$$

Thus the theorem follows from (3),(6),(8) and (10).

**Theorem 2.** Let $f$, $g$ and $h$ be three entire functions with $0 < L \lambda_f^h(p, q) \leq L \rho_f^h(p, q) < \infty$ and $0 < L \rho_f^h(m, q) < \infty$, where $p, q, m$ are positive integers with $q < \min\{p, m\}$. Then

$$\liminf_{r \to \infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{L \rho_f^h(p, q)}{L \rho_f^h(m, q)} \leq \limsup_{r \to \infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)}.$$

**Proof.** From the definition of relative $L - (p,q)$-th order we get for a sequence of values of $r$ tending to infinity,

$$\log^{[m]} G^{-1}H(r) \geq (L \rho_f^h(m, q) - \epsilon) \log^{[q]} [rL(r)].$$

Now from (9) and (11) it follows for a sequence of values of $r$ tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{L \rho_f^h(p, q)}{L \lambda_g^h(m, q)} + \epsilon.$$ 

As $\epsilon (> 0)$ is arbitrary we obtain that

$$\liminf_{r \to \infty} \frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \leq \frac{L \rho_f^h(p, q)}{L \lambda_g^h(m, q)}.$$ 

Again for a sequence of values of $r$ tending to infinity,

$$\log^{[p]} G^{-1}F(r) \geq (L \rho_f^h(p, q) - \epsilon) \log^{[q]} [rL(r)].$$

So combining (2) and (13) we get for a sequence of values of $r$ tending to infinity,

$$\frac{\log^{[p]} G^{-1}F(r)}{\log^{[m]} G^{-1}H(r)} \geq \frac{L \rho_f^h(p, q) - \epsilon}{L \lambda_g^h(m, q) + \epsilon}.$$
Since \( \epsilon ( > 0 ) \) is arbitrary it follows that

\[
\limsup_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)} \geq \frac{L \rho_g^f(p, q)}{L \rho_g^h(m, q)}.
\]  

(14)

Thus the theorem follows from (12) and (14).

The following theorem is a natural consequence of Theorem 1 and Theorem 2.

**Theorem 3.** Let \( f, g \) and \( h \) be three entire functions with \( 0 < L^* \lambda_g^f(p, q) \leq L \rho_g^f(p, q) < \infty \) and \( 0 < L^* \lambda_g^h(m, q) \leq L \rho_g^h(m, q) < \infty \) where \( p, q, m \) are positive integers with \( q < \min \{ p, m \} \). Then

\[
\liminf_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)} \leq \min \left\{ \frac{L \lambda_g^f(p, q)}{L \lambda_g^h(m, q)}, \frac{L \rho_g^f(p, q)}{L \rho_g^h(m, q)} \right\}
\]

\[
\leq \max \left\{ \frac{L \lambda_g^f(p, q)}{L \lambda_g^h(m, q)}, \frac{L \rho_g^f(p, q)}{L \rho_g^h(m, q)} \right\}
\]

\[
\leq \limsup_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)}.
\]

The proof is omitted.

In the following theorems we see some comparative growth properties of entire functions on the basis of relative \( L^* (p, q) \)th order and relative \( L^* - (p, q) \)th lower order where \( L = L(r) \) is a slowly changing function and \( p, q \) are positive integers with \( p > q \).

**Theorem 4.** Let \( f, g \) and \( h \) be three entire functions such that \( 0 < L^* \lambda_g^f(p, q) \leq L^* \rho_g^f(p, q) < \infty \) and \( 0 < L^* \lambda_g^h(m, q) \leq L^* \rho_g^h(m, q) < \infty \) where \( p, q, m \) are positive integers with \( q < \min \{ p, m \} \). Then

\[
\frac{L^* \lambda_g^f(p, q)}{L^* \rho_g^h(m, q)} \leq \liminf_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)} \leq \frac{L^* \lambda_g^f(p, q)}{L^* \lambda_g^h(m, q)}
\]

\[
\leq \limsup_{r \to \infty} \frac{\log^{[p]} G^{-1} F(r)}{\log^{[m]} G^{-1} H(r)} \leq \frac{L^* \rho_g^f(p, q)}{L^* \lambda_g^h(m, q)}.
\]

The proof is omitted because it can be carried out in the line of Theorem 1.
Theorem 5. Let $f$, $g$ and $h$ be three entire functions with $0 < L^* \lambda_g^f(p,q) \leq L^* \rho_g^f(p,q) < \infty$ and $0 < L^* \rho_g^h(m,q) < \infty$ where $p,q,m$ are positive integers with $q < \min \{p, m\}$. Then

$$\liminf_{r \to \infty} \frac{\log[G^{-1}F(r)]}{\log[G^{-1}H(r)]} \leq \frac{L^* \rho_g^f(p,q)}{L^* \rho_g^h(m,q)} \leq \limsup_{r \to \infty} \frac{\log[G^{-1}F(r)]}{\log[G^{-1}H(r)]}.$$ 

We omit the proof of Theorem 5 because it runs parallel to that of Theorem 2.

The following theorem is a natural consequence of Theorem 4 and Theorem 5.

Theorem 6. Let $f$, $g$ and $h$ be three entire functions such that $0 < L^* \lambda_g^f(p,q) \leq L^* \rho_g^f(p,q) < \infty$ and $0 < L^* \lambda_g^h(m,q) \leq L^* \rho_g^h(m,q) < \infty$ where $p,q,m$ are positive integers with $q < \min \{p, m\}$. Then

$$\liminf_{r \to \infty} \frac{\log[G^{-1}F(r)]}{\log[G^{-1}H(r)]} \leq \min \left\{ \frac{L^* \lambda_g^f(p,q)}{L^* \lambda_g^h(m,q)}, \frac{L^* \rho_g^f(p,q)}{L^* \rho_g^h(m,q)} \right\} \leq \limsup_{r \to \infty} \frac{\log[G^{-1}F(r)]}{\log[G^{-1}H(r)]}.$$ 

The proof is omitted.

References


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