Norms of Symmetrised Two-Sided Multiplication Operators

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Abstract
Let $P$ a projection and $Q_k$ defined from a self-adjoint operator $A_k$ be its canonical representation. We calculate the norm of the symmetrised two-sided multiplication operator $T_{PQ_k}E = PEQ_k + Q_kEP$ defined on a $C^*$-algebra $C^*(P,Q_k,1)$ generated by $P$ and $Q_k$ where $E$ is an idempotent related to $P$ and $Q_k$.

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1 Introduction
The knowledge of structural properties of the underlying $C^*$-algebra is undoubtedly one of the cornerstones in solving the problem of norms of elementary operators. Equivalently the spectral theorem helps a great deal in understanding Hilbert space theory. The spectral theory gives the unitary invariants of self-adjoint operator $A$ on a Hilbert space $H$ in terms of its multiplicity measure classes $\{\mu_n\}$ and provide unitarily equivalent model operators. Problems concerning pairs of projections play a fundamental role in the theory of operator algebras. A theorem by Pedersen [8], shows that if $P$ and $Q$ are projections then the $C^*$-algebra generated by $P$, $Q$ has a concrete realisation as an algebra of $2 \times 2$-matrix valued functions on $[0,1]$, so its representation theory is well understood. Given two pair of projections $P$ and $Q_k$ in terms of a generating self adjoint operator $A_k$, the $C^*$-algebra $A(E)$ generated by an idempotent $E$ is also generated by the range projections $P$, $Q_k$ of $E$ respectively $E^*$. The invariants of $P$, $Q_k$ are therefore unitary invariants as well. These to an extent
have been analysed also by [5]. In any $C^*$-algebra of operators say $\Omega$, we can define a symmetrised two-sided multiplication operator

$$X \mapsto T_{AB}X = AXB + BXA \quad A, B, X \in M(\Omega),$$

(1.1)

where $M(\Omega)$ is the multiplier algebra of $\Omega$. If $\Omega$ is a $C^*$-algebra of $n \times n$ matrices, then $M(\Omega) = \Omega$. The problem of calculating the norm of this operator $T_{AB}$ in equation (1.1) is still unresolved for a general Banach algebra. Several attempts, however, by various authors have been made for example see [3, 6, 7, 10, 11, 12]. The norm of this operator as revealed in many literatures is in the range of 1 and 2, that is, $\|A\|\|B\| \leq \|T_{AB} : B(H)\| \leq 2\|A\|\|B\|$. This implies, with right normalisation, that the optimal norm is 2. However, this maximal value has only been obtained in specific cases with certain restrictions. As our main result, we have obtain this maximal value in (1.1) for a $C^*$-algebra $A(P, Q_k, 1)$. These results are outlined in section four. Therefore, in this paper we have calculated the norm of the operator $T_{AB}$ defined on $C^*$-algebra $A(P, Q_k, 1)$ generated by $2 \times 2$ projection matrices $P$, $Q_k$ and 1, where $Q_k$ are determined by various self-adjoint operators $A_k$. How $A_k$ determines the representation of $Q_k$ will become clear in sections 2 and 3. Thus, we have shown that as $A_k \to 0$, $\|T_{AB}\| \to 2$ for a particular case of $A = P$, $B = Q_k$ and $A = 1 - 2P$, $1 - 2Q_k$. As usual in our workings, we will demand that $X$ be normalized. Although our results are not for a general Banach algebra, it adds some results to the special cases where the optimal upper bound can be obtained. The results are obtained by simple techniques. Our approach rely on the calculation of spectral radius and finally the use of limits. In section 2, we discussed the algebra of projections and section 3 the $C^*$-algebras generated by projections and their canonical representations and also those generated by idempotents. Section four contains our main results concerning the norm and Hilbert schmidt norm estimates of operator $T_{AB}$ and also in the same section, we have calculated the operator norm and the Hilbert-Schmidt norm of $T_{UV_k} + T_{PQ_k}$ with $U = 1 - 2P$, $V_k = 1 - 2Q_k$. A short discussion of the Fredholm properties of $T_{UV_k}E$, $T_{PQ_k}E$ and $T_{UV_k}E + T_{PQ_k}E$ are done in section 5. Thus, the conclusion in this section is that the Fredholm properties of these operators purely depends on the Fredholm properties of $A_k$. The sections of this paper are therefore organized as follows: section 1; Introduction, section 2; Projections, section 3; $C^*$-algebras, section 4; Symmetrised Two-Sided Multiplication Operators and finally section 5; Fredholm Properties.

2 Projections

Let $H$ be a Hilbert space and $B(H)$ denote the $C^*$-algebra of all bounded linear operators on $H$ and $\mathcal{F}(H) \subset B(H)$ the ideal of all finite rank operators.
A subalgebra $\mathcal{M} \subset B(H)$ is called a standard operator algebra on $H$ if it contains $\mathcal{F}(H)$. It is now clear that $\mathcal{F}(H) \subset \mathcal{M}, B(H)$. We can also define

$$\mathcal{M}_s = \{ A \in \mathcal{M} : A = A^* \}.$$ 

As $\mathcal{M}$ contains all finite rank operators, the set $\mathcal{M}_s$ contains all finite rank self adjoint operators. We can therefore define

$$\mathcal{F}_s(H) = \mathcal{F}(H) \cap \mathcal{M}_s = \mathcal{F}(H) \cap B_s(H)$$

to be the set of all self adjoint finite rank operators on $H$. We can assume that $\mathcal{M}_s = \mathcal{F}_s(H) = B_s(H)$. Therefore an equivalence relation on $\mathcal{F}_s(H)$ is defined as $A \equiv B$ if and only if $\text{Im} A \equiv \text{Im} B$. Here $\text{Im}$ denotes the image set of the operator. In each equivalence class there exist a unique projection $P$, which is the orthogonal projection on $\text{Im} A$. A projection of a finite rank can be identified with a finite dimensional subspace of $H$. Let $x, y \in H$ be unit vectors and $t, s$, arbitrary non-zero real numbers. If $xx^*$ denotes rank one projection onto the linear span of $x$, then it follows that $(txx^*)(syy^*)(txx^*) = 0$ if and only if $x$ and $y$ are orthogonal. Let two projections $P$ and $Q$ be defined by

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix},$$

where $\theta \in (0, \frac{\pi}{2})$, then $P$ and $Q$ are rank one projections and as $\theta \to \frac{\pi}{2}$, $P$ and $Q$ tend to be orthogonal. We also mention in passing that every rank one self adjoint operator is of the form $txx^*$ for some non zero real number $t$ and some unit vector $x$. It should be understood that this non zero real number $t$ depends on the operator itself. It is therefore straightforward to see that for every rank one operator $A \in \mathcal{F}_s(H)$, there exists some bounded linear operators $T_1, T_2 \in B(H)$ such that $T_1AT_2 = tA$ and $T_2AT_1 = sA$ for some non zero real numbers $s$ and $t$ depending on $A$. By the same argument, it can be shown that there is some positive real number $t$ such that for the same bounded linear operators $T_1$ and $T_2$ we have $T_2T_1AT_2T_1 = tA$ for every $A \in \mathcal{F}_s(H)$. For any pair of operators $A, B \in \mathcal{F}_s(H)$, we can choose any finite rank projection $P$ such that $PAP = A$ and $PB^*P = B$, then it will also follow that $PT_1AT_2P = T_1AT_2$ and $PT_1BT_2P = T_1BT_2$. In this case one can easily conclude that $A$ and $B$ are constant multiple of each other. It also follow by the same arguments that $T_2PT_1 = sP$ for some non zero real number $s$ depending on $P$. It is worth noting here that all these triple products are possible because the $C^*$-algebras $B(H)$ is closed under Jordan triple product. Therefore, we can find some operator $T \in B(H)$ such that for $T^* \in B(H)$, we have $PTAT^*P = sP$ and $PTBT^*P = tP$ for some constants $s$ and $t$.

As an example let $B(H) = M_2(\mathbb{C})$ and choose $P = \text{diag}(1, 0), A = \text{diag}(q, 0), B = \text{diag}(a, 0)$, then if $T = (r_{ij}), \ (i = 1, 2), a, q \in \mathbb{R}$ and $r_{ij} \in \mathbb{C}$ we have $s = |r_{11}|^2 \ q, t = |r_{11}|^2 \ a$ and $c = \frac{a}{s}$ for $A = cB$. 


Now let \( u \) and \( v \) be real positive numbers. Then there exist unique positive real numbers \( \alpha \) and \( \beta \in (0, 1) \) such that \( u + v = \alpha^2 \), \( u = \beta \alpha^2 \) and \( v = (1 - \beta) \alpha^2 \). Therefore we can find rank one projections \( P \), \( Q \), and \( A \) such that \( PQ = QP = 0 \), \( APA = \beta A \) and \( AQA = (1 - \beta) A \). If we denote \( R \) by \( R = P + Q \), then \( R \) becomes a rank two projection and \( T_1 RT_2 = T_1 PT_2 + T_1 QT_2 \). To show this claim, we only need to find \( 2 \times 2 \) matrix projections of rank one as stated earlier on that satisfy the above conditions and elementary matrix computations show that \( A = \text{diag}(1,0) \),

\[
P = \begin{bmatrix} \beta & \eta \\ \eta & 1 - \beta \end{bmatrix}, \quad Q = \begin{bmatrix} 1 - \beta & -\eta \\ -\eta & \beta \end{bmatrix}
\]

where \( \eta = (\beta(1 - \beta))^{\frac{1}{2}} \). With the assumption that \( \beta = \sin^2 A_k \) for some \( A_k \in (0, \frac{\pi}{2}) \), then the matrices \( P \) and \( Q \) above can be computed explicitly in terms of \( A_k \). It is from this background, with this kind of understanding that we develope the norm theory of symmetrised two-sided multiplication operators defined on \( C^* \)-algebras generated by two projections and an idempotent all depending on a self adjoint operator defined from the angle between \( \text{Ran} P \) and \( \text{Ran} Q_k \). In developing our \( C^* \)-algebra in the next section, we will therefore fix the projection \( P \) and vary \( Q_k \) over \( A_k \).

### 3 \( C^* \)-algebras

In this section we only highlight the results that the author and Behncke [2] had obtained. These will be stated without proofs. The proofs can be obtained from [2, 5] and the literature stated therein. The \( C^* \)-algebras generated by \( P \), \( Q_k \) and \( 1 \) will form underlying spaces on which the operator \( T_{PQ_k} \) will be defined.

An operator \( A \) on a Hilbert space \( \mathcal{H} \) will be called to be of type \( I_n \), if \( \mathcal{A}(A, 1) \) only has irreducible representations of dimensions less than \( n \), i.e. \( \hat{A} \equiv_n \hat{A} \) [[4], ch. 3.6]. Selfadjoint operators are thus of type \( I_1 \). In this case there is a unitary map: \( U : \mathcal{H} \to \bigoplus \mathcal{H}_i \otimes \mathbb{C}^i \) so that \( A \) becomes the sum of its “simple” multiplicity components

\[
UAU^{-1} = \bigoplus (A_i \otimes 1_i).
\]

Now let \( P \) and \( Q_k \) be two orthogonal projections. Then it is well known that \( \mathcal{A} = \mathcal{A}(P, Q_k, 1) \) is of type \( I_2 \), that is, irreducible representations of \( \mathcal{A} \) are either one or two dimensional. Moreover the generic two dimensional irreducible representation has the form

\[
P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_k = \begin{pmatrix} c^2 & sc \\ sc & s^2 \end{pmatrix}, \quad s = \sin \beta, c = \cos \beta,
\]

(3.2)
where 0 < \beta < \frac{\pi}{2}. From this the following result can easily be deduced by standard \( C^* \)-algebra and von Neumann algebra techniques.

**Theorem 3.1** Let \( P \) and \( Q_k \) be two orthogonal projections on the Hilbert space \( \mathcal{H} \). Then \( \mathcal{H} \) has a unique decomposition \( \mathcal{H} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \oplus \mathcal{H}_2) \) so that

(i) \( \mathcal{H}_1 = \{ x \in \mathcal{H} | PQ_kx = Q_kPx \} \) is a reducing subspace for \( P \) and \( Q_k \).

(ii) \( P = P_1 + P_2 \), \( Q_k = Q_{k1} + Q_{k2} \) where \( P_1 = P \mid \mathcal{H}_1 \), \( P_2 = P \mid \mathcal{H}_1^\perp \), \( Q_{k1} = Q_k \mid \mathcal{H}_1 \) and \( Q_{k2} = Q_k \mid \mathcal{H}_1^\perp \). \( \mathcal{H}_2 = P_2 \mid \mathcal{H}_1^\perp \).

(iii) There exists a selfadjoint operator \( A_k \) on \( \mathcal{H}_2 \) with \( \sigma(A_k) \subset [0, \frac{\pi}{2}] \), and \( 0, \frac{\pi}{2} \notin \sigma_p(A_k) \) so that \( P \) and \( Q_k \) are unitarily equivalent to

\[
P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_{k2} = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}, \quad c = \cos A_k, s = \sin A_k
\]

\[ (3.3) \]

(iv) The unitary invariants of \( P \) and \( Q_k \) are those of \( A \) and the multiplicities

\[
\dim(\mathcal{H}_1 \oplus (P_1 + Q_{k1} - P_1Q_{k1})\mathcal{H}_1), \quad \dim(P_1 - P_1Q_{k1}), \quad \dim(Q_{k1} - P_1Q_{k1}), \quad \dim(P_1Q_{k1}).
\]

\[ (3.4) \]

In the remainder, Theorem 3.1 will be used systematically to obtain information about operators \( A_k \) for which \( P \) and \( Q_k \) are the range projections of \( A_k \) respectively \( A_k^* \). As a rule, statements about the commutative parts \( P_1 \), \( Q_{k1} \) tend to be trivial.

Let \( E \) be an idempotent on the Hilbert space \( H \) and let \( P \) and \( Q_k \) be the range projections of \( E \) respectively \( E^* \), i.e. \( PEE^* = EE^*P \) and \( Q_kE^*E = E^*EQ_k \). Then \((1 - P)E(I - P)E)^* = 0 \) shows \( PE = E \) and by symmetry

\[
PEQ_k = E
\]

\[ (3.5) \]

As before write \( \mathcal{H} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \oplus \mathcal{H}_2) \) and \( P = P_1 + P_2 \), \( Q_k = Q_{k1} + Q_{k2} \) for the decomposition of \( \mathcal{H} \), \( P \) and \( Q_k \) into its commutative and genuinely type \( I_2 \) parts.

**Theorem 3.2**

a) A pair of projections \( P, Q_k \) is the range projections of \( E \) respectively \( E^* \) iff \( P_1Q_{k1} = P_1 = Q_{k1} \) and if the operator \( A_k \) of Theorem 3.1 satisfies \( \| A_k \| < \frac{\pi}{2} \).

b) \( \mathcal{A}(P, Q_k, 1) = \mathcal{A}(E, 1) \) and the unitary invariants of \( E \) are those of \( P, Q_k \), i.e. \( \dim P_1Q_{k1}, \dim \mathcal{H}_1 - \dim P_1Q_{k1} \) and those of \( A_k \).
c) Any idempotent \( E \) on \( \mathcal{H} \) is unitarily equivalent to an operator of the form

\[
E = 0 \oplus 1 \oplus \begin{pmatrix} 1 & tgA_k \\ 0 & 0 \end{pmatrix}
\]
on \((\mathcal{H}_{10} \oplus \mathcal{H}_{11}) \oplus (\mathcal{H}_2 \oplus \mathcal{H}_2)\)  \hspace{1cm} (3.6)

where \( \mathcal{H}_1 = \mathcal{H}_{10} \oplus \mathcal{H}_{11} \) and where \( A_k \) satisfies \( A_k^* = A_k \) and \( ||A_k|| < \frac{\pi}{2} \) with \( 0 \not\in \sigma_p(A_k) \).

One can thus show easily from (3.6) using spectral mapping theorem that

\[
||E||^2 = ||EE^*|| = ||1 + tg^2A_k|| = \frac{1}{\cos^2 ||A_k||} \hspace{1cm} (3.7)
\]

and \( \sigma(A_k) \subset [0, \beta], \beta < \frac{\pi}{2} \). Theorem 3.2 thus shows that the idempotent \( E \) is similar to the projection \( P_1 + P_2 \). By induction one can then show that a finite family of orthogonal idempotents is similar to a finite family of orthogonal projections.

Now fix \( P \) and vary \( Q_k \) over \( A_k \) and define an idempotent \( E \) which is related to \( P \) and \( Q_k \). A symmetrised two-sided multiplication operator can thus be defined on the \( C^* \)-algebras \( \mathcal{A}(P, Q_k, 1) = \mathcal{A}(E, 1) \). For simplicity let

\[
P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_k = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & \frac{s}{c} \\ 0 & 0 \end{bmatrix}.
\]

4 Symmetrised Two-sided Multiplication Operators

Now define a symmetrised two-sided multiplication operator \( T_{AB} \) as in equation (1.1), with \( A = P \), \( B = Q_k \) and without loss of generality take \( X = E \) since \( X \) will be normalized in order to calculate \( ||T_{PQ_k}|| \). To normalize \( E \), we will tend \( A_k \) to zero. Since \( \mathcal{A}(P, Q_k, 1) = \mathcal{A}(E, 1) \) from results of Theorem 3.2 and \( \mathcal{A}(P, Q_k, 1) = M_2(\mathbb{C}) \), this algebra will be the same as its multiplier algebra. Thus,

**Theorem 4.1** Let \( T_{PQ_k}E = PEQ_k + Q_kEP \) be a symmetrised two-sided multiplication operator, then \( \lim_{A_k \to 0} ||T_{PQ_k}|| = 2 \) and \( \lim_{A_k \to 0} ||T_{PQ_k}||_2 = 2 \)

**Proof** A simple matrix computation show that

\[
T_{PQ_k}(E) = PEQ_k + Q_kEP = E + Q_kP.
\]

Thus the eigenvalues of \( T_{PQ_k}E \) are given by

\[
\lambda_{1/2} = \frac{1}{2} \{(1 + c_k^2) \pm \sqrt{(1 + c_k^2)^2 + 4s_k^2}\}.
\]
As $A_k \to 0$, (3.7) implies that $\|E\| \to 1$ and by spectral radius formula, one has

$$\|TPQ_k\| = \sup_{\|E\| \to 1} \|TPQ_k(E)\| \geq \sup_{\|E\| \to 1} \{|\lambda_i| : \lambda_i \in \sigma(TPQ_k E)\} = 2$$

With the right normalisation, it is easy to show that $\|TPQ_k\| \leq 2$ as $A_k \to 0$. To prove the second claim, use the following explicit formula for calculating Hilbert Schmidt norm

$$\|\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}\|^2 = \frac{1}{2} (|a|^2 + |b|^2 + |c|^2 + \sqrt{(|a|^2 + |b|^2 + |c|^2)^2}).$$

Thus one has

$$\|TPQ_k(E)\|^2 = \frac{1 + 3c_k^4}{c_k^2}.$$ 

The desired results now follow at once. □

**Corollary 4.2** Let $P$ and $Q_k$ be projections with $P = PQ_k = Q_k$, then there exists a unitary operator $W_k$ with $W_kPW_k^{-1} = Q_k$ and $W_kQ_kW_k^{-1} = P$ and $W_kEW_k^{-1} = E^*$ for the idempotent $E$ associated to $P$ and $Q_k$, then $\|TW_kW_k^*\| \to 2$ as $A_k \to 0$.

**Proof** By functional calculus, it suffices to solve this on $\mathbb{C}^2$. Here $W_k$ is just the reflection on the line halving the angle between $PH$ and $Q_kH$. Explicitly $W_k$ is given by $W_k = \begin{bmatrix} -c & -s \\ -s & c \end{bmatrix}$. These are the main results of [1]. It follows by direct matrix calculation that $W_kEW_k^{-1} = W_kEW_k^* = E^*$. The last claim now follows from Theorem 4.1 and the fact that $W_k = W_k^{-1} = W_k^*$. □

**Remark 4.3** The unitary equivalence problem for operators on a Hilbert space has received a lot of attention over the years. The analogue problem for operators on a Banach space, due to lack of an inner product structure, requires different techniques directly related to the specific setting under consideration. Therefore given a symmetric norm ideal $\mathfrak{I}$ of $B(H)$ which is different from $C_2(H)$ and fixed elements $A_n$ and $B_n$ ($n = 1, 2$) in $B(H)$, we can let $\delta_n : \mathfrak{I} \to \mathfrak{I}$ be defined by $\delta_n = A_nEB_n$. Then $\delta_1$ is isometrically equivalent to $\delta_2$ if and only if there exist unitary operators $W_1, W_2 \in B(H)$ such that for some $\lambda \in \mathbb{C}$ we have

$$A_2 = \lambda W_1A_1W_1^*, \quad B_2 = \frac{1}{\lambda} W_2B_1W_2^*.$$ 

In this setting, the summands of $TPQ_kE$ can be split such that $\delta_1(E) = PEQ_k$ and $\delta_2(E) = Q_kEP$. With idempotent $E$ being symmetric and making the
right substitutions, we have $W_1 = \begin{bmatrix} -c & -s \\ -s & c \end{bmatrix}$ and $W_2 = \begin{bmatrix} c & s \\ s & -c \end{bmatrix}$ and $\lambda = 1$. Note that as $A_k \to 0$, not only is $\delta_1$ and $\delta_2$ isometrically equivalent but they also tend to be unitarily equivalent.

**Theorem 4.4** Let $U = 1 - 2P$ and $V_k = 1 - 2Q_k$ be unitaries of order 2. As $A_k \to 0$, we have $\|T_{UV_k}\| \to 2$ and also $\|T_{UV_k}\|_2 \to 2$.

**Proof** A quick check shows that $Q_k E = Q_k$, $E Q_k = E$, $PE = E$, $EP = P$ and therefore

$$T_{UV_k}(E) = UEV_k + V_k EU = 2E - 2P - 2Q_k + 4Q_kP.$$ 

The proofs follow immediately from Theorem 4.1.

Now define an operator

$$T_{A_1B_1}(X) + T_{A_2B_2}(X) = \sum_{i=1}^{2} A_i X B_i + B_i X A_i, \quad (4.1)$$

where $A_2, B_2$ somehow are related to $A_1$ and $B_1$ respectively in the underlying $C^*$-algebra. For the sake of this study, call the operator in (4.1) a double symmetrised two-sided multiplication operator. Define the operator in (4.1) with the following assumptions: $A_1 = P$, $B_1 = Q_k$, $A_2 = 1 - 2P$, $B_2 = 1 - 2Q_k$ and $X = E$. Then (4.1) reduces to

$$(T_{UV_k} + T_{PQ_k})(E) = UEV_k + V_k EU + PEQ_k + Q_k EP. \quad (4.2)$$

**Theorem 4.5** Assume (4.2) holds, then

(i) $\lim_{A_k \to 0} \|T_{UV_k} + T_{PQ_k}\| = 4$

(ii) $\lim_{A_k \to 0} \|T_{UV_k} + T_{PQ_k}\|_2 = 4$.

**Proof**

(i) It follows that $(T_{UV_k} + T_{PQ_k})(E) = 3E - 2P - 2Q_k$ and the eigenvalues of $(T_{UV_k} + T_{PQ_k})(E)$ are given by

$$\lambda_1 = \frac{1}{2} \left\{ \gamma + (\gamma^2 - \beta) \frac{1}{2} \right\}, \quad \lambda_2 = \frac{1}{2} \left\{ \gamma - (\gamma^2 - \beta) \frac{1}{2} \right\},$$

where $\gamma = 4c^2 - s^2$, $\beta = 8s^2$. As before we obtain

$$\|T_{UV_k} + T_{PQ_k}\| \geq \frac{1}{2} \sup_{\|E\| \to 1} \left\{ \| \gamma + (\gamma^2 - \beta) \frac{1}{2} \|, \| \gamma - (\gamma^2 - \beta) \frac{1}{2} \| \right\}.$$ 

Similar arguments to those of Theorem 4.1 clears the proof.
(ii) Given any operator $A$, its Hilbert-Schmidt norm $\|A\|_2$ is defined by $\|A\|_2 = (\text{tr} |A|^2)^{1/2}$. This is easily calculated from eigenvalues. Therefore, with the results in part (i), we have

$$\|(T_{UV_k} + T_{PQ_k})(E)\|_2 = \left\{ \sum_{i=1}^{2} |\lambda_i|^2 \right\}^{1/2} \|E\||E\| = \frac{1}{\sqrt{2}} \left\{ 2\gamma^2 - \beta \right\}^{1/2}$$

and thus $\lim_{A_k \to 0} \|(T_{UV_k} + T_{PQ_k})\|_2 = 4$.

5 Fredholm Properties

**Theorem 5.1** The operators $T_{PQ_k}E$, $T_{UV_k}(E)$, and $(T_{UV_k} + T_{PQ_k})(E)$ are Fredholm iff $A_k$ is.

**Proof** To show that $T_{PQ_k}E$ is Fredholm, we need to show that the matrix representation of $T_{PQ_k}E$ is invertible and this is so if and only if $-\sin^2 A_k$ is invertible, that is, if $A_k$ is invertible. Therefore $T_{PQ_k}E$ is Fredholm if and only if $A_k$ is. Similarly $T_{UV_k}(E)$ is invertible if and only if $-2\sin^2 A_k$ is invertible, that is, if $A_k$ is invertible. Hence the implication. Finally $(T_{UV_k} + T_{PQ_k})(E)$ is invertible iff $-11\sin^2 A_k$ is invertible implying that only if $A_k$ is invertible. This completes the proof and hence it becomes clear that the Fredholmness of these operators depends on that of $A_k$.

**Remark 5.2** As proposed in [12] an application of essential numerical range can provide an alternative proof for Theorem 4.1 provided $W_{m,e}(P^*, Q_k^*) \cap W_{m,e}(P, Q_k) \neq \emptyset$. With the assumption that this holds one has $\|T_{PQ_k}\|_{cb} = \|T_{PQ_k}\| = \|PP^*\| + \|Q_kQ_k\|$ where the subscript $cb$ denotes completely bounded norm. As $A_k \to 0$, note that $\|P\| = \|Q_k\| = 1$.

**References**


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