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Lebesgue Constants for Fourier Expansions Associated with Weight Function $(1-x)^\alpha(1+x)^\beta + M\delta_{-1} + N\delta_1$

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Abstract

Let $d\mu(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^\alpha(1+x)^\beta dx + M\delta_{-1} + N\delta_1$, $\alpha, \beta \geq -1/2$ and $\gamma = \max\{\alpha,\beta\} > -1/2$, be a measure on$[-1,1]$, where δ_t is the delta function at a point t. In this paper we investigate the behaviour of the Lebesgue constants $||S_n f||_p = \sup\{||S_n f||_{L^p(d\mu)} : f \in L^p(d\mu), ||f||_{L^p(d\mu)} \leq 1\}$ for $p \notin (\frac{4\gamma+4}{2\gamma+3}, \frac{4\gamma+4}{2\gamma+1})$, where $S_n f$ is the nth partial sum of the Fourier expansion of a function f in terms of the polynomials orthogonal with respect to the measure μ.

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1 Introduction and Main Result

Let $\omega_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $(\alpha, \beta > -1)$, be the Jacobi weight on the interval $[-1,1]$, and let denote by $\{p_n^{(\alpha,\beta,M,N)}\}_{n=0}^\infty$ the sequence of orthonormal polynomials with respect to the measure

$$d\mu(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\omega_{\alpha,\beta}(x)dx + M\delta_{-1} + N\delta_1,$$

where $\alpha > -1$, $\beta > -1$, and $M, N \geq 0$. These polynomials were introduced by Koornwinder in [10] and constitute a natural generalization of so-called Krall's Jacobi type polynomials earlier introduced by Krall in [11], which are orthogonal with respect to the measure $((1-x)^\alpha + M\delta(x))\,dx$ on $[0,1]$. We call these polynomials Koornwinder's Jacobi type polynomials. In [10] the author established diferent properties of the Koornwinder's Jacobi type polynomials such as the differential equation that they satisfy, a representation as a hypergeometric series. In [1], [12], [13] the authors looked for higher order of differential equations satisfied by the Koornwinder's Jacobi type polynomials, and in [9] the authors derived the distribution and three-term recurrence relation for such polynomials. However, the asymptotic properties of Koornwinder's Jacobi type polynomials were not known until its consideration by one of the authors in [4], [5].

We shall say that $f(x) \in L^p(d\mu)$ if $f(x)$ is μ-measurable on the $[-1,1]$ and $||f||_{L^p(d\mu)} < \infty$, where

$$||f||_{L^p(d\mu)} = \begin{cases} \left(\int_{-1}^1 |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} & if\ 1 \le p < \infty, \\ ess\ sup_{-1<x<1} |f(x)| & if\ p = \infty. \end{cases}$$

For $f \in L^1(d\mu)$, let $S_n f$ denote the nth partial sum of the Fourier expansion of f in Koornwinder's Jacobi-type polynomials, that is,

$$S_n f(x) = \sum_{k=0}^{n} \hat{f}(k) p_k^{(\alpha,\beta,M,N)}(x)$$

where the Fourier coefficients are

$$\begin{aligned} \hat{f}(k) &= \int_{-1}^1 f(x) p_k^{(\alpha,\beta,M,N)}(x) d\mu(x) \\ &= \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^1 f(x) p_k^{(\alpha,\beta,M,N)}(x) \omega_{\alpha,\beta}(x) dx \\ &+ M f(-1) p_k^{(\alpha,\beta,M,N)}(-1) + N f(1) p_k^{(\alpha,\beta,M,N)}(1). \end{aligned}$$

The study of the convergence of $S_n f$ in $L^p(d\mu)$, $1 \le p \le \infty$, has been discussed in [4], [5], [6], [8]). If $\alpha \ge \beta \ge -1/2$ and $\alpha > -1/2$, then (see [6])

$$||S_n f||_{L^p(d\mu)} \le C ||f||_{L^p(d\mu)} \qquad \forall n \ge 0,\ \forall f \in L^p(d\mu)$$

if and only if p belongs to the Pollard interval (p_0, q_0), where

$$p_0 = \frac{4(\alpha+1)}{2\alpha+3}, \qquad q_0 = \frac{4(\alpha+1)}{2\alpha+1}.$$

The aim of this paper is to study the Lebesgue constants

$$||S_n f||_p = \sup\{||S_n f||_{L^p(d\mu)} : f \in L^p(d\mu), ||f||_{L^p(d\mu)} \le 1\}$$

for $p \notin (p_0, q_0)$. By duality, it suffices to assume that $q_0 \le p \le \infty$. For $M = N = 0$ the estimate of the Lebesgue constants has been derived in [2], [14],

[15], and note that the sharp estimates for $p \in \{p_0, q_0\}$ are given in early paper [2] than in [14], [15].

Throughout this paper positive constants are denoted by $c,\ c_1, ...$ and they may vary at every occurrence. The notation $u_n \cong v_n$ means that the sequence u_n/v_n converges to 1 and notation $u_n \sim v_n$ means $c_1 u_n \leq v_n \leq c_2 u_n$ for sufficiently large n.

Our main result is:

Theorem 1.1 *Let $\alpha \geq \beta \geq -1/2$ and $\alpha > -1/2$. Then*

$$c(\log n)^{1/p} \leq ||S_n f||_p \leq c \log n, \qquad p = q_0,$$

$$||S_n f||_p \sim n^{\alpha+1/2-(2\alpha+2)/p}, \qquad q_0 < p \leq \infty.$$

The left-hand inequality in theorem is a special case of a result given in [4]. More precisely, the following proposition holds:

Proposition 1.2 *Let $\alpha \geq \beta \geq -1/2$ and $\alpha > -1/2$. There exists a positive constant c, independent of n, such that*

$$||S_n f||_p \geq c \begin{cases} (\log\ n)^{\frac{1}{p}}\ if\ p = q_0, \\ n^{\frac{2\alpha+1}{2} - \frac{2\alpha+2}{p}}\ if\ q_0 < p \leq \infty. \end{cases}$$

Proof. We need only to prove for $M > 0$, $N > 0$ and $p = \infty$. Taking into account in [4, Proposition 2.3] and [3, formula (5)], we have that

$$ess\ sup_{-1<x<1}\ |p_n^{(\alpha,\beta,M,N)}(x)| = cn^{\alpha+1/2}.$$

Now, the result follows from [4, formula (3.12)].

Remark 1.3 *Using the simmetry formula in [10], for the case $M = 0$ and $N > 0$ we get the same results as in the case $M > 0$ and $N = 0$ but exchanging α and β. Therefore, we will consider only the cases when $M > 0$ and $N \geq 0$.*

2 Proof of Theorem 1.1

Let $\alpha \geq \beta \geq -1/2$, $\alpha > -1/2$ and $q_0 \leq p < \infty$. For $f \in L^p(d\mu)$ (see [6])

$$\begin{aligned} ||S_n f||^p_{L^p(d\mu)} &\leq cM^p|f(-1)|^p \int_{-1}^1 |L_n^{(\alpha,\beta,M,N)}(x,-1)|^p \omega_{\alpha,\beta}(x)dx \\ &+ cN^p|f(1)|^p \int_{-1}^1 |L_n^{(\alpha,\beta,M,N)}(x,1)|^p \omega_{\alpha,\beta}(x)dx \\ &+ \int_{-1}^1 |\int_{-1}^1 L_n^{(\alpha,\beta,M,N)}(x,y)\ f(y)\ \omega_{\alpha,\beta}(y)dy|^p \omega_{\alpha,\beta}(x)dx \\ &+ M|S_n f(-1)|^p + N|S_n f(1)|^p, \end{aligned} \tag{1}$$

where $L_n^{(\alpha,\beta,M,N)}(x,y)=\sum_{k=0}^{n} p_k^{(\alpha,\beta,M,N)}(x)\ p_k^{(\alpha,\beta,M,N)}(y)$ and

$$|S_n f(-1)| \le c, \quad |S_n f(1)| \le c. \tag{2}$$

For estimate the first and second term in (1), we call the following proposition (see [7, Proposition 8]):

Proposition 2.1 *Let* $\theta \in [0,\pi]$.

(a) For $M>0$

$$|L_n^{(\alpha,\beta,M,N)}(\cos\theta,-1)| \le c \begin{cases} \theta^{-\alpha-1/2} \ if \ 1/n \le \theta \le \pi, \\ n^{\alpha+1/2} \ if \ 0 \le \theta \le 1/n. \end{cases}$$

(b) For $N>0$

$$|L_n^{(\alpha,\beta,M,N)}(\cos\theta,1)| \le c \begin{cases} \theta^{-\beta-1/2} \ if \ 0 \le \theta \le \pi-1/n, \\ n^{\beta+1/2} \ if \ \pi-1/n \le \theta \le \pi. \end{cases}$$

As to the first term of (1), Proposition 2.1 yields

$$\begin{aligned} &\int_0^{\pi} |L_n^{(\alpha,\beta,M,N)}(cos\theta,-1)|^p \theta^{2\alpha+1}(\pi-\theta)^{2\beta+1}(x)d\theta \\ \le\ & cn^{p\alpha+p/2}\int_0^{n^{-1}} \theta^{2\alpha+1}d\theta + c\int_{n^{-1}}^{\pi/2} \theta^{2\alpha+1}\theta^{-p\alpha-p/2}d\theta \\ +\ & c\int_{\pi/2}^{\pi}(\pi-\theta)^{2\alpha+1}d\theta \le c \begin{cases} n^{p\alpha+p/2-2\alpha-2} \ if \ p>q_0, \\ \log\ n \ if \ p=q_0. \end{cases} \end{aligned} \tag{3}$$

In a similar way

$$\begin{aligned} &\int_0^{\pi} |L_n^{(\alpha,\beta,M,N)}(cos\theta,1)|^p \ \theta^{2\alpha+1}(\pi-\theta)^{2\beta+1}(x)d\theta \\ \le\ & c \begin{cases} n^{p\beta+p/2-2\beta-2} \ if \ p>\frac{4\beta+4}{2\beta+1}, \\ \log\ n \ if \ p=\frac{4\beta+4}{2\beta+1}, \\ 1 \ if \ p<\frac{4\beta+4}{2\beta+1} \end{cases} \le c \begin{cases} n^{p\alpha+p/2-2\alpha-2} \ if \ p>q_0, \\ \log n \ if \ p=q_0. \end{cases} \end{aligned} \tag{4}$$

The basic tool in the estimate of the third term of (1) is Pollard's decomposition of the kernel $L_n^{(\alpha,\beta,M,N)}(x,y)$ in the form (see [17], [6])

$$L_n^{(\alpha,\beta,M,N)}(x,y) = r_n T_1(n,x,y) + s_n T_2(n,x,y) - s_n T_3(n,x,y)$$

where $\{r_n\}$, $\{s_n\}$ are bounded sequences and

$$T_1(n,x,y) = p_{n+1}^{(\alpha,\beta,M,N)}(x)\ p_{n+1}^{(\alpha,\beta,M,N)}(y),$$

$$T_2(n,x,y) = (1-x^2)\frac{p_n^{(\alpha+1,\beta+1)}(x)\ p_{n+1}^{(\alpha,\beta,M,N)}(y)}{x-y},$$

$$T_3(n,x,y) = (1-y^2)\frac{p_n^{(\alpha+1,\beta+1)}(y)\ p_{n+1}^{(\alpha,\beta,M,N)}(x)}{x-y}.$$

Let

$$W_{i,n}(f,x) = \int_{-1}^{1} T_i(n,x,y)f(y)\omega_{\alpha,\beta}(y)dy, \qquad i = 1,2,3.$$

For $i = 1$, by using Hölder's inequality and [3, Theorem 1], we have

$$\begin{aligned} |W_{1,n}(f,x)| &= \left|p_{n+1}^{(\alpha,\beta,M,N)}(x)\int_{-1}^{1} f(y)p_{n+1}^{(\alpha,\beta,M,N)}(y)\omega_{\alpha,\beta}(y)dy\right| \\ &\leq c\left|p_{n+1}^{(\alpha,\beta,M,N)}(x)\right| ||f||_{L^p(\omega_{\alpha,\beta}(\cdot)d\cdot)}. \end{aligned}$$

Hence

$$\int_{-1}^{1} |W_{1,n}(f,x)|^p\omega_{\alpha,\beta}(x)dx \leq c\begin{cases} n^{p\alpha+p/2-2\alpha-2} \ if\ q_0 < p < \infty, \\ \log\ n\ if\ p = q_0. \end{cases}$$

For $i = 2,3$, from [3, formula (5)] and [18, Theorem 7.32.2] and by using the same technique to the one used in [2], we have

$$\int_{-1}^{1} |W_{i,n}(f,x)|^p\omega_{\alpha,\beta}(x)dx \leq c\begin{cases} n^{p\alpha+p/2-2\alpha-2} \ if\ q_0 < p < \infty, \\ (\log\ n)^p\ if\ p = q_0. \end{cases}$$

Therefore

$$\int_{-1}^{1} |\int_{-1}^{1} L_n^{(\alpha,\beta,M,N)}(x,y)\ f(y)\ \omega_{\alpha,\beta}(y)dy|^p\omega_{\alpha,\beta}(x)dx \leq c\begin{cases} n^{p\alpha+p/2-2\alpha-2} \ if\ q_0 < p < \infty, \\ (\log\ n)^p\ if\ p = q_0. \end{cases} \tag{5}$$

By using (2), (3), (4) and (5) we find from (1) that

$$||S_nf||_{L^p(d\mu)} \leq \begin{cases} n^{\alpha+1/2-2(\alpha+1)/p} \ if\ q_0 < p < \infty, \\ \log\ n\ if\ p = q_0. \end{cases}$$

Now let us consider the case when $N = 0$ and $p = \infty$. For $f \in L^\infty(d\mu)$

$$\begin{aligned} |S_nf(x)| &\leq \int_{-1}^{1} |f(y)\ L_n^{(\alpha,\beta,M,0)}(x,y)|d\mu(y) \leq c\int_{-1}^{1} |L_n^{(\alpha,\beta,M,0)}(x,y)| \\ &\times\ \omega_{\alpha,\beta}(y)dy + M|f(-1)|\ |L_n^{(\alpha,\beta,M,0)}(x,-1)|. \end{aligned} \tag{6}$$

The following simple proposition provides an important relation betwen $L_n^{(\alpha,\beta,M,0)}(x,y)$ and the kernel $K_n^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{n} p_k^{(\alpha,\beta)}(x)p_k^{(\alpha,\beta)}(y)$, and some their properties.

Proposition 2.2 *The following estimates hold:*

a) $L_n^{(\alpha,\beta,M,0)}(x,y) = K_n^{(\alpha,\beta)}(x,y) - \frac{MK_n^{(\alpha,\beta)}(x,-1)}{1+MK_n^{(\alpha,\beta)}(-1,-1)} K_n^{(\alpha,\beta)}(-1,y)$,

b) $K_n^{(\alpha,\beta)}(-1,-1) \sim n^{2\beta+2}$,

c) $K_n^{(\alpha,\beta)}(-1,y) \sim (-1)^n n^{\beta+1/2} p_n^{(\alpha,\beta+1)}(y)$,

d) $L_n^{(\alpha,\beta,M,0)}(x,-1) = \frac{K_n^{(\alpha,\beta)}(x,-1)}{1+MK_n^{(\alpha,\beta)}(-1,-1)} \sim (-1)^n n^{-\beta-3/2} p_n^{(\alpha,\beta+1)}(x)$.

Proof. The formulae *a*) is given in the proof of Proposition 5 in [7]. The part *b*) follows from equations (4.1.3) and (4.5.8) in [18]. The part *c*) follows from equations (4.1.3), (4.3.4) and (4.5.3) in [18]. Finally, *a*), *b*) and *c*) yields *d*).

Applying Proposition 2.2 in (6), we obtain

$$\begin{aligned} |S_n f(x)| &\leq c\int_{-1}^{1} |K_n^{(\alpha,\beta)}(x,y)|\, \omega_{\alpha,\beta}(y)dy \\ &+ cn^{-1}|p_n^{(\alpha,\beta+1)}(x)| \int_{-1}^{1} |p_n^{(\alpha,\beta+1)}(y)|\omega_{\alpha,\beta}(y)dy + cn^{\alpha+1/2}. \qquad (7) \end{aligned}$$

The first term of (7) is (see [16], [15])

$$\int_{-1}^{1} |K_n^{(\alpha,\beta)}(x,y)|\, \omega_{\alpha,\beta}(y)dy \leq cn^{\alpha+1/2}$$

In order to estimate the second term of (7), we will distinguish the following two cases:

I. Let $-1/2 \leq \beta \leq \alpha \leq \beta+1$ and $\alpha > -1/2$. From [18, Theorem 7.32.1] and [18, Theorem 7.34] we obtain

$$\begin{aligned} n^{-1}|p_n^{(\alpha,\beta+1)}(x)| \int_{-1}^{1} |p_n^{(\alpha,\beta+1)}(y)|\omega_{\alpha,\beta}(y)dy &\leq c\frac{n^{\alpha+1/2}}{n^{\alpha-\beta}} \int_{-1}^{1} |p_n^{(\alpha,\beta+1)}(y)|\omega_{\alpha,\beta}(y)dy \\ \leq c\frac{n^{\alpha+1/2}}{n^{\alpha-\beta}} \int_{0}^{1} |p_n^{(\alpha,\beta+1)}(y)|(1-y)^{\alpha}dy &+ c\frac{n^{\alpha+1/2}}{n^{\alpha-\beta}} \int_{-1}^{0} |p_n^{(\alpha,\beta+1)}(y)|(1+y)^{\beta}dy \\ &\leq cn^{\alpha+1/2} + c\frac{n^{\alpha+1/2}}{n^{\alpha-\beta}} \int_{-1}^{0} |p_n^{(\alpha,\beta+1)}(y)|(1+y)^{\beta}dy. \end{aligned}$$

If $\beta = -1/2$, then $\alpha > \beta$ and

$$\frac{1}{n^{\alpha-\beta}} \int_{-1}^{0} |p_n^{(\alpha,\beta+1)}(y)|(1+y)^{\beta}dy \leq \frac{\log\ n}{n^{\alpha-\beta}} \leq c.$$

If $\beta > -1/2$, then we have

$$\int_{-1}^{0} |p_n^{(\alpha,\beta+1)}(y)|(1+y)^{\beta}dy \leq c.$$

II. Let $\alpha > \beta + 1$. Again, as applications of [18, Theorem 7.32.1] and [18, Theorem 7.34] we have

$$n^{-1}|p_n^{(\alpha,\beta+1)}(x)| \int_{-1}^{1} |p_n^{(\alpha,\beta+1)}(y)|\omega_{\alpha,\beta}(y)dy \le c\frac{n^{\alpha+1/2}}{n} \int_{0}^{1} |p_n^{(\alpha,\beta+1)}(y)| \times(1-y)^{\alpha}dy + c\frac{n^{\alpha+1/2}}{n} \int_{-1}^{0} |p_n^{(\alpha,\beta+1)}(y)|(1+y)^{\beta}dy \le cn^{\alpha+1/2}.$$

As a conclusion

$$||S_n f||_{L^{\infty}(d\mu)} \le cn^{\alpha+1/2}$$

in the case when $N = 0$.

Finally, let us consider the case when $M, N > 0$ and $p = \infty$. First we prove the following proposition:

Proposition 2.3 *The following estimates hold*

(a) $L_n^{(\alpha,\beta,M,N)}(x,y) = L_n^{(\alpha,\beta,M,0)}(x,y) - \frac{NL_n^{(\alpha,\beta,M,0)}(x,1)}{1+NL_n^{(\alpha,\beta,M,0)}(1,1)} L_n^{(\alpha,\beta,M,0)}(1,y),$

(b) $\left|\frac{NL_n^{(\alpha,\beta,M,0)}(x,1)}{1+NL_n^{(\alpha,\beta,M,0)}(1,1)}\right| \le c.$

Proof. The formulae (a) is given in the proof of Proposition 5 in [7]. (b) From Proposition 2.2 a), [18, formula (4.5.3)] and [18, Theorem 7.32.1]

$$\begin{aligned} \left|L_n^{(\alpha,\beta,M,0)}(x,1)\right| &\le \left|K_n^{(\alpha,\beta)}(x,1)\right| + \left|\frac{MK_n^{(\alpha,\beta)}(-1,1)}{1+MK_n^{(\alpha,\beta)}(-1,-1)}\right| \left|K_n^{(\alpha,\beta)}(x,-1)\right| \\ &\le cn^{2\alpha+2} + c\frac{n^{\alpha+\beta+1}}{n^{2\beta+2}} n^{\alpha+\beta+2} \le cn^{2\alpha+2}. \end{aligned}$$

Now, taking into account that (see [4])

$$L_n^{(\alpha,\beta,M,0)}(1,1) \cong cn^{2\alpha+2}$$

the result follows.

From Proposition 2.1 and Proposition 2.3, for $f \in L^{\infty}(d\mu)$

$$\begin{aligned} |S_n f(x)| &\le \int_{-1}^{1} |f(y)\ L_n^{(\alpha,\beta,M,N)}(x,y)| d\mu(y) \\ &\le c\int_{-1}^{1} |L_n^{(\alpha,\beta,M,N)}(x,y)|\omega_{\alpha,\beta}(y)dy + M|f(-1)||L_n^{(\alpha,\beta,M,N)}(x,-1)| \\ &+ N|f(1)|\ |L_n^{(\alpha,\beta,M,N)}(x,1)| \le c\int_{-1}^{1} |L_n^{(\alpha,\beta,M,0)}(x,y)|\ \omega_{\alpha,\beta}(y)dy \\ &+ c\int_{-1}^{1} |L_n^{(\alpha,\beta,M,0)}(1,y)|\ \omega_{\alpha,\beta}(y)dy + cn^{\alpha+1/2}. \end{aligned} \quad (8)$$

We have showed above that

$$\int_{-1}^{1} |L_n^{(\alpha,\beta,M,0)}(x,y)|\ \omega_{\alpha,\beta}(y)dy \leq cn^{\alpha+1/2}. \tag{9}$$

On the other hand, from Proposition 2.2, [18, formula (4.5.3)] and [18, Theorem 7.34]

$$\int_{-1}^{1} |L_n^{(\alpha,\beta,M,0)}(1,y)|\ \omega_{\alpha,\beta}(y)dy \leq c\int_{-1}^{1} |K_n^{(\alpha,\beta)}(1,y)|\ \omega_{\alpha,\beta}(y)dy$$
$$+cn^{\alpha-\beta-1}\int_{-1}^{1} |K_n^{(\alpha,\beta)}(-1,y)|\ \omega_{\alpha,\beta}(y)dy \leq cn^{\alpha+1/2}\int_{-1}^{1} |p_n^{(\alpha+1,\beta)}(y)|\ \omega_{\alpha,\beta}(y)dy$$
$$+cn^{\alpha-1/2}\int_{-1}^{1} |p_n^{(\alpha,\beta+1)}(y)|\ \omega_{\alpha,\beta}(y)dy \leq cn^{\alpha+1/2}. \tag{10}$$

Now, from (9) and (10) we find from (8) that

$$||S_n f||_{L^\infty(d\mu)} \leq cn^{\alpha+1/2}$$

in the case when $M, N > 0$.

The Theorem 1.1 is proved.

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