Neumann Problem for Coupled Diffusion Systems
with Localized Nonlinear Reactions

Wanjuan Du 1, Zhongping Li and Li Xie

College of Mathematics and Information
China West Normal University
Nanchong 637002, P.R. China

Abstract

This paper deals with the Neumann problem for coupled diffusion systems with localized nonlinear reactions. We give the blow-up conditions and the asymptotic behavior of the blow-up solution.

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1 Introduction and main results

We consider positive solutions to the system of diffusion equations coupled with localized source

\[ u_{it} = \Delta u_i + u_{i+1}^p(x_0,t) \quad (i = 1, 2, \ldots, k), \quad u_{k+1} := u_1, \quad x \in \Omega, \quad t > 0, \quad (1.1) \]

with homogeneous Neumann boundary values

\[ \frac{\partial u_i}{\partial \nu} = 0 \quad (i = 1, 2, \ldots, k), \quad x \in \partial \Omega, \quad t > 0, \quad (1.2) \]

and nontrivial, nonnegative and bounded initial data

\[ u_i(x, 0) = u_{i0}(x) \quad (i = 1, 2, \ldots, k), \quad x \in \Omega, \quad t > 0, \quad (1.3) \]

where \( p_i > 0 \) (\( i = 1, 2, \ldots, k \)). \( \Omega \in \mathbb{R}^N \) is a bounded domain with smooth boundary \( \partial \Omega \), and \( \nu \) is the outward unit normal vector on the boundary \( \partial \Omega \).

Equations (1.1) describe a physical phenomenon where the reaction in a dynamical system is driven by the temperature at a single point (see [1, 4, 7]).

1duwanjuan28@163.com
The uncoupled single equation of (1.1) and some its variants were studied by some authors (see [2, 5]). In [2], Chadam et al. studied the single more general equation
\[ u_t = \Delta u + f(u(x_0(t), t)), \quad x \in \Omega, \; t > 0, \]
with Neumann boundary conditions. They gave the blow-up conditions, and proved that the blow-up set is the whole region \( \overline{\Omega} \) if a solution blows up at finite time \( T \). They also showed that \( u(x_0, t) \leq \left( \frac{2}{(y-1)(T-t)} \right)^{\frac{1}{p-1}} \) for \( f(s) = s^p, (p > 1) \).

We remark that a lot of work have been done in the past few years on the blow-up problems for coupled systems (see [3, 6-9]). In the special cases \( k = 2 \) of (1.1)-(1.3), the blow-up rate and blow-up set were studied in [9]. There are some works on the blow-up rate for a general semilinear diffusion system
\[ u_{it} = \Delta u_i + u_{i+1}^{p_i} \quad (i = 1, 2, \ldots k), \quad u_{k+1} := u_1, \quad (x, t) \in \Omega \times (0, +\infty) \]
with homogeneous Dirichlet boundary conditions, where \( \Omega \in \mathbb{R}^N \) or \( \Omega = \mathbb{R}^N \) (see [3, 8] and references therein).

Motivated by the above mentioned works, the aim of this paper is to present the asymptotic behavior of the blow-up solution to the system (1.1)-(1.3). Now, we introduce some useful symbols to state our results. Let \( \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_k)^T \) be the solution of the following linear algebraic system
\[
A\alpha := \begin{pmatrix}
1 & -p_1 & 0 & \ldots & 0 & 0 \\
0 & 1 & -p_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & -p_{k-1} \\
-p_k & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{k-1} \\
\alpha_k
\end{pmatrix}
= \begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix}.
\tag{1.4}
\]

A series of standard computations yield \( det A = 1 - \prod_{i=1}^{k} p_i \). We shall see that \( det A = 0 \) is the critical global existence curve. A direct computation also shows that
\[
\alpha_i = \frac{1 + p_i + \sum_{j=i+1}^{k} p_i \ldots p_j}{\prod_{i=1}^{k} p_i - 1} \quad (i = 1, 2, \ldots, k).
\tag{1.5}
\]

From (1.4), we see that
\[
\alpha_i + 1 = p_i \alpha_{i+1} \quad (i = 1, 2, \ldots, k), \quad \alpha_{k+1} := \alpha_1.
\tag{1.6}
\]

Now we state the main results of this paper.

**Theorem 1.1** (i) If \( \prod_{i=1}^{k} p_i \leq 1 \) (i.e. \( det A \geq 0 \)), then every solution of the system (1.1)-(1.3) is global in time; (ii) If \( \prod_{i=1}^{k} p_i > 1 \) (i.e. \( det A < 0 \)), then all solutions of (1.1)-(1.3) blow up in finite time.
Theorem 1.2 Let \((u_1, u_2, ..., u_k)\) be the solution of (1.1)-(1.3) in \(\Omega \times (0, T)\), which blows up in finite time \(T\). Then there exist positive constants \(C_i\) such that \(\lim_{t \to T} u_i(x, t)(T-t)^{\alpha_i} \leq C_i\) \((i = 1, 2, ..., k)\), \((x, t) \in \Omega \times (0, T)\), uniformly in \(\Omega\).

Finally, we give a brief line of the rest of this paper. In Section 2, we give the blow-up conditions and prove Theorem 1.1. The proof of Theorem 1.2 is the subject of Section 3.

2 Blow up in finite time

In this section, we characterize when all solutions to the problem (1.1)-(1.3) are global in time or blow up. Now, we start our arguments with the maximum principle that will be used in the sequel.

Lemma 2.1 Let \((u_1, u_2, ..., u_k)\) be a classical solution of the problem

\[
\begin{align*}
 u_{it} - \Delta u_i &\geq c_i(x,t)u_{i+1}(x,t) \quad (i = 1, 2, ..., k), \quad u_{k+1} := u_1, \quad (x, t) \in \Omega \times (0, T), \\
 \frac{\partial u_i}{\partial n} &= 0 \quad (i = 1, 2, ..., k), \quad (x, t) \in \partial \Omega \times (0, T), \\
 u_i(x, 0) &\geq 0 \quad (i = 1, 2, ..., k), \quad x \in \Omega.
\end{align*}
\]
(2.1)

If \(0 \leq c_i(x, t) < C_i\), then \(u_i(x, t) \geq 0 \quad (i = 1, 2, ..., k)\), for all \((x, t) \in \overline{\Omega} \times [0, T]\).

Proof. Set \(w_i = e^{-Kt}u_i\), where \(K = \sum_{i=1}^{k} C_i\). We claim \(w_i \geq 0\) on \(\overline{\Omega} \times [0, T']\) for any \(T' < T\). In fact, if \(\min(w_1, w_2, ..., w_k)(\overline{\Omega}, \overline{T}) < 0\) for some \((\overline{x}, \overline{t}) \in \overline{\Omega} \times [0, T']\), without loss of generality, we assume that \(\min(w_1, w_2, ..., w_k)(x, t)\) takes negative minimum at \((\overline{x}, \overline{t})\) and \(w_1(\overline{x}, \overline{t}) \leq w_i(\overline{x}, \overline{t}), i = 2, ..., k\). Using the first inequality in (2.1), we find that

\[
w_{1t} - \Delta w_1 \geq -K w_1(x, t) + c_1(x, t)w_2(x, t), \quad (x, t) \in \Omega \times [0, T'].
\]
(2.2)

If \((\overline{x}, \overline{t}) \in \Omega \times (0, T']\), then we have

\[
w_{1t}(\overline{x}, \overline{t}) - \Delta w_1(\overline{x}, \overline{t}) \geq -K w_1(\overline{x}, \overline{t}) + c_1(\overline{x}, \overline{t})w_2(x, \overline{t}) \geq -K w_1(\overline{x}, \overline{t}) + C_1 w_1(\overline{x}, \overline{t}) > 0,
\]
(2.3)

here we use \(0 \leq c_1(\overline{x}, \overline{t}) \leq C_1\) and \(w_2(x, \overline{t}) \geq \min(w_1, w_2, ..., w_k)(x, \overline{t}) \geq \min(w_1, w_2, ..., w_k)(\overline{x}, \overline{t}) = w_1(\overline{x}, \overline{t})\). On the other hand, \(w_1(x, t)\) attains negative minimum at \((\overline{x}, \overline{t})\), so \(w_{1t}(\overline{x}, \overline{t}) - \Delta w_1(\overline{x}, \overline{t}) \leq 0\), which leads to a contradiction to inequality (2.3).

If \((\overline{x}, \overline{t}) \in \partial \Omega \times (0, T']\), we have \(w_{1t}(\overline{x}, \overline{t}) = 0\). In this case, we may choose a small \(\epsilon > 0\) satisfying \(\epsilon < -(K - C_1)w_1(\overline{x}, \overline{t})\), and find a point \(x_\epsilon \in \Omega\), sufficiently close to \(x\), such that

\[
w_1(x_\epsilon, \overline{t}) \leq w_1(\overline{x}, \overline{t}) + \frac{\epsilon}{3K}, \quad w_{1t}(x_\epsilon, \overline{t}) \leq \frac{\epsilon}{3}, \quad -\Delta w_1(x_\epsilon, \overline{t}) \leq \frac{\epsilon}{3}.
\]
Then combining these inequalities with (2.2), we obtain \( c_1(x, \overline{t})w_2(x_0, \overline{t}) \leq w_1(x, \overline{t}) - \Delta w_1(x, \overline{t}) + K w_1(x, \overline{t}) \leq \epsilon + K w_1(x, \overline{t}) \). It follows that \( \epsilon \geq -K w_1(x, \overline{t}) + c_1(x, \overline{t})w_2(x_0, \overline{t}) \geq - (K - C_1) w_1(x, \overline{t}) \), which contradicts our choice of \( \epsilon \). Thus, \( \min(w_1, w_2, ..., w_k) \geq 0 \) on \( \overline{\Omega} \times [0, T) \) and \( u_i(x, t) \geq 0 \) (\( i = 1, 2, ..., k \)) on \( \overline{\Omega} \times [0, T) \).

**Proof of Theorem 1.1(i).** For \( \prod_{i=1}^{k} p_i < 1 \), we follow from (1.5) that \( -\alpha_i > 1 \). Construct \( \overline{u}_i(x, t) = (M + t)^{-\alpha_i} \) (\( i = 1, 2, ..., k \)), where \( M > 0 \) is to be determined later. Define \( \overline{u}_{i+1} = \overline{u}_1 \) and \( \alpha_{k+1} = \alpha_1 \). It will be obtained from (1.6) that \( \overline{u}_i t - \Delta \overline{u}_i = -\alpha_i (M + t)^{-\alpha_i - 1} > (M + t)^{-p_i \alpha_i + 1} = \overline{u}_{i+1}^\alpha \) (\( i = 1, 2, ..., k \)). It is easy to see that \( \frac{\partial \overline{u}_i}{\partial \nu} = 0 \), \( x \in \partial \Omega \), \( t \geq 0 \), and \( \overline{u}_i(x, 0) \geq u_{i0}(x) \) (\( i = 1, 2, ..., k \)), where we take \( M \) sufficiently large. It follows from Lemma 2.1 that \( (\overline{u}_1, \overline{u}_2, ..., \overline{u}_k) \geq (u_1, u_2, ..., u_k) \), which implies \( (u_1, u_2, ..., u_k) \) globally exists.

For \( \prod_{i=1}^{k} p_i = 1 \), set \( \overline{u}_i = Ae^{-Lt} \) (\( i = 1, 2, ..., k \)), where \( A > 0 \), \( L_i \) are to be determined. A simple computation shows \( \overline{u}_i t - \Delta \overline{u}_i = AL_i e^{-Lt} \geq A\rho_i e^{p_i L_i + 1} = \overline{u}_{i+1}^\alpha \) (\( i = 1, 2, ..., k \)), where \( \overline{u}_{k+1} := \overline{u}_1 \), \( L_{k+1} := L_1 \), if we choose \( L_i \) large enough and \( L_i = p_i L_{i+1} \). In the case of \( i = 1 \), we must confirm \( L_1 = p_1 L_2 = p_1 p_2 L_3 = ... = L_1 \prod_{i=1}^{k} p_i \). Noting \( \frac{\partial \overline{u}_i}{\partial \nu} = 0 \), \( x \in \partial \Omega \), \( t \geq 0 \), and \( \overline{u}_i(x, 0) \geq u_{i0}(x) \) (\( i = 1, 2, ..., k \)) for \( A \) sufficiently large, by Lemma 2.1, we have \( (\overline{u}_1, \overline{u}_2, ..., \overline{u}_k) \geq (u_1, u_2, ..., u_k) \), which implies \( (u_1, u_2, ..., u_k) \) globally exists. We have proved Theorem 1.1(i) for system (1.1)-(1.3).

**Proof of Theorem 1.1(ii).** Since the initial data are nontrivial, without loss of generality we assume that \( u_{i0}(x) \geq 0 \) and \( u_{i0}(x) \neq 0 \). Let \( v \) be the solution of \( v_t - \Delta v = 0 \) in \( \Omega \times (0, \infty) \) with \( v(x, 0) = u_{i0}(x) \). It follows from the maximum principle for the heat equation that \( u_1 \geq v \) as long as \( u_1 \) exists. Moreover, \( v(x, t) > 0 \) in \( \Omega \times (0, \infty) \). On the other hand, \( u_k \) satisfies \( u_{kt} - \Delta u_k = u_{tk}^\alpha(x_0, t) \geq v_{tk}^\alpha(x_0, t) > 0 \) in \( \Omega \times (0, T) \) with \( u_{k0}(x) \geq 0 \). So \( u_k(x, t) > 0 \) in \( \Omega \times (0, T) \). Similarly, we have \( u_i(x, t) > 0 \) (\( i = 2, 3, ..., k - 1 \)) in \( \Omega \times (0, T) \). Therefore, without loss of generality, we may assume \( u_{i0}(x) > 0 \) (\( i = 1, 2, ..., k \)) for \( x \in \overline{\Omega} \).

Now, we prove the non-existence of global solutions by constructing a blow-up subsolution of the system (1.1)-(1.3). Consider the ODE system

\[
\begin{align*}
    f_i'(t) &= f_i^{p_i}(t) \quad (i = 1, 2, ..., k), \\
    f_k(0) &= a_i > 0,
\end{align*}
\]

where \( a_i = \min_{\overline{\Omega}} u_{i0}(x) \). By Lemma 2.1, we have \( (f_1, f_2, ..., f_k) \geq (u_1, u_2, ..., u_k) \).

We claim that exist a constants \( \delta > 0 \) such that \( \left( \prod_{i=1}^{k} f_i \right)'(t) \geq \delta \left( \prod_{i=1}^{k} f_i(t) \right)^m \), where \( m = 1 + \frac{1}{\sum \alpha_i} \) (the claim is to be proved later).

For \( \prod_{i=1}^{k} p_i > 1 \), (1.5) implies that \( m > 1 \). Noting \( \left( \prod_{i=1}^{k} f_i \right)(0) = \prod_{i=1}^{k} a_i > 0 \), it follow that \( \left( \prod_{i=1}^{k} f_i \right)'(t) \) blows up in finite time. So dose \( (u_1, ..., u_k) \), which implies that Theorem 1.1(ii) holds.
Now we prove the claim. We denote first by
\[
\bar{A} := \begin{pmatrix}
p_1 & -1 & 0 & \ldots & 0 & 0 \\
0 & p_2 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & p_{k-1} & -1 \\
-1 & 0 & 0 & \ldots & 0 & p_k \\
\end{pmatrix}.
\]
and then observe that the linear algebraic system \( \bar{A}(\frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_{k-1}}, \frac{1}{q_k})^T = (m - 1, m - 1, \ldots, m - 1, m - 1)^T \) exists a unique solution
\( q_1 = m - 1 \) if and only if \( q_i > 1 \) (i = 1, 2, ..., k). After a series of computations, we get \( q_i > 1 \) which imply that \( 0 < \frac{1}{q_i} < 1 \) or equivalently \( q_i > 1 \). Then we use Hölder’s inequality to obtain
\[
\left( \prod_{i=1}^{k} f_i(t) \right)^{m} = f_2^{p_1+1} f_3^{p_2+1} f_4^{p_3+1} \ldots f_k^{p_k+1} \leq C f_2^{p_1+1} f_3^{p_2+1} f_4^{p_3+1} \ldots f_k^{p_k+1}
\]
Combining the above inequality with \( f_i'(t) = f_{i+1}'(t) \) \( i = 1, 2, ..., k \), \( f_{k+1} := f_1 \), we have proved our claim. □

3 Asymptotic behavior

In this section, we concern with the asymptotic behavior of blow-up solution near the blow-up time and the set of blow-up points. We first give an important lemma.

Lemma 3.1 Let \( w \in C^{2,1}(\overline{\Omega} \times (0, T)) \) be a solution of the problem
\[
\begin{align*}
\frac{\partial w}{\partial t} - \Delta w &= g(t), & (x, t) \in \Omega \times (0, T), \\
\frac{\partial w}{\partial \nu} &= 0, & (x, t) \in \partial \Omega \times (0, T), \\
w(x, 0) &= w_0(x) \geq 0, & x \in \Omega.
\end{align*}
\]
Then we have \( \lim_{t \to T} \|w(\cdot, t)\|_{\infty} = +\infty \) if and only if \( \int_0^T g(s)ds = +\infty \). Furthermore, if \( \lim_{t \to T} \|w(\cdot, t)\|_{\infty} = +\infty \), then \( \lim_{t \to T} \frac{w(x, t)}{G(t)} = \lim_{t \to T} \frac{w(x, t)}{G(t)} = 1 \), uniformly in \( \overline{\Omega} \), where \( G(t) = \int_0^t g(s)ds \).

Proof. Let \( G(x, y; t - \tau) \) be Green’s function associated with the heat operator \( \frac{\partial}{\partial t} - \Delta \). Note that the function \( G(x, y; t - \tau) \) possesses the properties \( G(x, y; t - \tau) \geq 0 \), \( \int_{\Omega} G(x, y; t)dy = 1 \). Then we have
\[
w(x, t) = \int_{\Omega} G(x, y; t)w_0(y)dy + \int_0^t \int_{\Omega} G(x, y; t - \tau)g(\tau)dyd\tau \\
= \int_{\Omega} G(x, y; t)w_0(y)dy + \int_0^t g(\tau)d\tau, \quad (x, t) \in \Omega \times (0, T).
\]
It follows that \( \int_0^t g(\tau) d\tau \le w(x,t) \le \|w_0\|_\infty + \int_0^t g(\tau) d\tau, \quad (x,t) \in \Omega \times (0,T). \)
From the inequality, we see that Lemma 3.1 holds. \( \square \)

**Lemma 3.2** Let \( (u_1,u_2,...,u_k) \) be a solution to the problem (1.1)-(1.3). If \( (u_1,u_2,...,u_k) \) blows up in finite time \( T \), then \( u_i \ (i = 1,2,...,k) \) blow up simultaneously.

**Proof.** Suppose on the contrary that \( u_i \ (i = 1,2,...,k) \) do not blow up simultaneously in finite time \( T \). Without loss of generality, we may assume that \( u_1 \) blows up in finite time \( T \) and \( u_2 \) is bounded on \( \overline{\Omega} \times [0,T] \). That is, there exists a constant \( C > 0 \) such that \( 0 \le u_2 \le C \) for all \( (x,t) \in \overline{\Omega} \times [0,T] \). From the Lemma 3.1, we know that \( u_1 \) is bounded on \( \overline{\Omega} \times [0,T] \), which is a contradiction. \( \square \)

**Remark.** Lemmas 3.1 and 3.2 imply that the blow-up set of a blow-up solution of (1.1)-(1.3) is the whole region \( \overline{\Omega} \).

In the following, we intend to prove the result of Theorems 1.2. For convenience, from now on we write \( g_i(t) = u_{i+1}^{p_i}(x_0,t), \ G_i(t) = \int_0^t g_i(s) ds \ (i = 1,2,...,k), \ u_{k+1} := u_1, \ G_{k+1} := G_1(t). \)

**Lemma 3.3** If \( (u_1,u_2,...,u_k) \) is the solution of (1.1)-(1.3) with blow-up time \( T \), then there exist positive constants \( C_i \) such that \( \lim_{t \to T} G_i(t)(T-t)^{\alpha_i} \le C_i \ (i = 1,2,...,k) \) uniformly in \( \overline{\Omega} \).

**Proof.** By similar arguments in the proof of Theorem 1.1(ii), we have \( \left( \prod_{i=1}^k G_i(t) \right) \sim (t \to T), \ \lambda \left( \prod_{i=1}^k G_i(t) \right)^{1+\sum_{i=1}^k \alpha_i}, \) as \( t \to T \), where \( \lambda \) is a positive constant. We get by integrating

\[
\prod_{i=1}^k G_i(t) \le C(T-t)^{-\sum_{i=1}^k \alpha_i} \quad \text{as } t \to T. \quad (3.1)
\]

With (3.1) at hand, we claim there exist \( C_i > 0 \) such that \( G_i(t) \le C_i(T-t)^{-\alpha_i} \) \( (i = 1,2,...,k) \), as \( t \to T \). In fact, we only need to show the case \( i = 1 \).

On the contrary, noting \( G_i(t) \sim G_{i+1}(t) \), if there exist two sequences \( \{t_n\} \), \( (0 < t_n < T) \), and \( \{c_n\} \) with \( t_n \to T^- \) and \( c_n \to \infty \) as \( n \to \infty \) such that

\[
G_1(t_n) \ge c_n(T-t_n)^{-\alpha_1}, \quad \text{for large } n,
\]

we have

\[
G_k(t) \ge G_k(t_n) + c \int_{t_n}^t G_k^{p_k}(s) ds \ge cG_k^{p_k}(t_n)(t-t_n) \ge cc_n^{p_k}(T-t_n)^{-p_k\alpha_1}(t-t_n),
\]

\[
G_{k-1}(t) \ge G_{k-1}(t_n) + c \int_{t_n}^t G_{k-1}(s) ds \ge c^{1+p_{k-1}}(T-t_n)^{-p_{k-1}\alpha_1}(t-t_n) \int_{t_n}^t (s-t_n)^{p_k-1} ds
\]

\[
= \frac{1}{p_{k-1}+1}c^{1+p_{k-1}}(T-t_n)^{-p_{k-1}\alpha_1}(t-t_n)^{p_{k-1}+1},
\]

\[
G_1(t) \ge c(p_1,...,p_{k-1})c_n^{p_1}(T-t_n)^{-\alpha_1} \prod_{i=1}^{k-1} p_i (t-t_n)^{1+\sum_{i=1}^{k-1} \Pi_{i=1}^i p_i}
\]

\[
\square
\]
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where $c$ is a suitable constant and $e(p_1, ..., p_{k-1})$ is a constant depending only on $p_i$ ($i = 1, 2, ..., k - 1$). Multiplying the above inequalities, we get

$$
\prod_{i=1}^{k} G_i(t) \geq e()^{n}(T - t_n)^{-p_{\alpha_1}}(t - t_n)^q,
$$

(3.2)

where $e, \theta > 0$ are constants depending only on $p_i$, and

$$
p = \sum_{i=0}^{k-1} \prod_{i=k-1}^{k} p_i, \quad q = 1 + \left\{ 1 + p_{k-1} \right\} + ... + \left\{ p_1 p_2 ... p_{k-1} + p_1 p_2 ... p_{k-2} + ... + p_1 + 1 \right\}.
$$

Taking $t = t' := \frac{T + t_n}{2}$ in (3.2), we have $t' \to T$ as $n \to \infty$ and

$$
\prod_{i=1}^{k} G_i(t') \geq e()^{n}2^{-q} (T - t_n)^{-p_{\alpha_1} + q} \geq e()^{n}2^{-p_{\alpha_1}} (T - t')^{-p_{\alpha_1} + q}.
$$

(3.3)

Using the definitions of $\alpha_i$; $p$; $q$, after a series of complicated but standard computations, we have $-p_{\alpha_1} + q = -\sum_{i=1}^{k} \alpha_i$, which implies inequality (3.3) contradicts (3.1) since $e()^{n} \to \infty$ as $n \to \infty$. So the claim $G_1(t) \leq C_1(T - t)^{-\alpha_1}$ as $t \to T$ is true. Proceeding in a similar way with the other (in)equalities, we conclude $G_i(t) \leq C_i(T - t)^{-\alpha_1}$ ($i = 2, ..., k$), as $t \to T$.

We obtain the conclusion of the Theorem 1.2 by combining Lemmas 3.1 and 3.3.

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References


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