On Some Further Results of Growth Properties of Composite Entire and Meromorphic Functions

Sanjib Kumar Datta

Department of Mathematics, University of North Bengal
P.O. North Bengal University, Raja Rammohunpur
Dist-Darjeeling, PIN-734013, West Bengal, India
sanjib.kr_datta@yahoo.co.in

Tanmay Biswas

Department of Mathematics, University of North Bengal
P.O. North Bengal University, Raja Rammohunpur
Dist-Darjeeling, PIN-734013, West Bengal, India
Tanmaybiswas_math@rediffmail.com

Abstract

In this paper we study the comparative growth properties of composite entire and meromorphic functions considering left factor or right factor to be of order zero.

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1 Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let $f$ be a meromorphic function and $g$ be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [5] and [12]. In the sequel we use the following notation:

$\log^k x = \log \left( \log^{k-1} x \right)$ for $k = 1, 2, 3, \ldots$ and $\log^0 x = x$.

We now recall the following definitions:
**Definition 1** The order $\rho_f$ and the lower order $\lambda_f$ of a meromorphic function $f$ are defined as

$$
\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.
$$

If $f$ is an entire function, one can easily verify that

$$
\rho_f = \limsup_{r \to \infty} \frac{\log^2 M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^2 M(r, f)}{\log r}.
$$

**Definition 2** The type $\sigma_f$ of a meromorphic function $f$ is defined as

$$
\sigma_f = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.
$$

If $f$ is entire, then

$$
\sigma_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.
$$

**Definition 3** A function $\rho_f(r)$ is called a proximate order of $f$ relative to $T(r, f)$ if (i) $\rho_f(r)$ is non-negative and continuous for $r \geq r_0$, say, (ii) $\rho_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\rho'_f(r - 0)$ and $\rho'_f(r + 0)$ exist, (iii) $\lim \rho_f(r) = \rho_f < \infty$, (iv) $\lim r \rho'_f(r) \log r = 0$ and (v) $\limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho'_f(r)}} = 1$.

**Definition 4** A function $\lambda_f(r)$ is called a lower proximate order of $f$ relative to $T(r, f)$ if (i) $\lambda_f(r)$ is non-negative and continuous for $r \geq r_0$, say, (ii) $\lambda_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda'_f(r - 0)$ and $\lambda'_f(r + 0)$ exist, (iii) $\lim \lambda_f(r) = \lambda_f < \infty$, (iv) $\lim r \lambda'_f(r) \log r = 0$ and (v) $\liminf_{r \to \infty} \frac{T(r, f)}{r^{\lambda'_f(r)}} = 1$.

If $\rho_f < \infty$ then $f$ is of finite order. Also $\rho_f = 0$ means that $f$ is of order zero. In this connection Liao and Yang [8] gave the following definition.

**Definition 5** [8]Let $f$ be a meromorphic function of order zero. Then the quantities $\rho_f^*$ and $\lambda_f^*$ of a meromorphic function $f$ are defined as :

$$
\rho_f^* = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log^2 r} \quad \text{and} \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log^2 r}.
$$

If $f$ is an entire function then clearly

$$
\rho_f^* = \limsup_{r \to \infty} \frac{\log^2 M(r, f)}{\log^2 r} \quad \text{and} \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log^2 M(r, f)}{\log^2 r}.
$$
Datta and Biswas [3] gave an alternative definition of zero order and zero lower order of a meromorphic function which is the following.

**Definition 6 [3]** Let \( f \) be a meromorphic function of order zero. Then the quantities \( \rho_f^* \) and \( \lambda_f^* \) of \( f \) are defined by:

\[
\rho_f^* = \limsup_{r \to \infty} \frac{T(r,f)}{\log r} \quad \text{and} \quad \lambda_f^* = \liminf_{r \to \infty} \frac{T(r,f)}{\log r}.
\]

For entire \( f \),

\[
\rho_f^* = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r} \quad \text{and} \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log M(r,f)}{\log r}.
\]

In this paper we investigate some growth properties of composite entire and meromorphic functions of order zero.

## 2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1** [2] If \( f \) and \( g \) are two entire functions then for all sufficiently large values of \( r \),

\[
M\left(\frac{1}{8} M\left(\frac{r}{2},g\right) - |g(0)|, f\right) \leq M(r,f \circ g) \leq M (M(r,g), f).
\]

**Lemma 2** [11] Let \( f \) be entire and \( g \) be a transcendental entire function of finite lower order. Then for any \( \delta > 0 \),

\[
M(r^{1+\delta}, f \circ g) \geq M (M(r,g), f) \quad (r \geq r_0).
\]

**Lemma 3** [1] If \( f \) be meromorphic and \( g \) be entire then for all sufficiently large values of \( r \),

\[
T(r,f \circ g) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T (M(r,g), f).
\]

**Lemma 4** [6] Let \( g \) be an entire function with \( \lambda_g < \infty \) and assume that \( a_i (i = 1, 2, ..., n; n \leq \infty) \) are entire functions satisfying \( T(r,a_i) = o\{T(r,g)\}. If \)

\[
\sum_{i=1}^{n} \delta(a_i, g) = 1, then \lim_{r \to \infty} \frac{T(r,g)}{\log M(r,g)} = \frac{1}{\pi}.
\]
Lemma 5 \[7\] If \( f \) be an entire function, then for \( \delta > 0 \) the function \( r^{\rho_f + \delta - \rho_f(r)} \) is ultimately an increasing function of \( r \).

Lemma 6 Let \( f \) be an entire function. Then for \( \delta > 0 \) the function \( r^{\lambda_f + \delta - \lambda_f(r)} \) is ultimately an increasing function of \( r \).

Proof. Since

\[
\frac{d}{dr} r^{\lambda_f + \delta - \lambda_f(r)} = \{\lambda_f + \delta - \lambda_f(r) - r\lambda_f'(r) \log r\} r^{\lambda_f + \delta - \lambda_f(r) - 1} > 0
\]

for all sufficiently large values of \( r \), the lemma is proved. \( \blacksquare \)

Lemma 7 \[4\] Let \( f \) be a meromorphic function and \( g \) be transcendental entire. If \( \rho_{fog} < \infty \), then \( \rho_f = 0 \).

Lemma 8 \[3\] Let \( f \) be meromorphic and \( g \) be entire such that \( \rho_f < \infty \) and \( \rho_g = 0 \). Also let \( g \) be transcendental entire. Then \( \rho_{fog} < \infty \).

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 If \( f \) be any meromorphic function of order zero. Then (i) \( \rho_f^* = 1 \) and (ii) \( \lambda_f^* = 1 \).

Proof. From the definition of \( \rho_f^{**} \) and \( \lambda_f^{**} \) we have for arbitrary positive \( \varepsilon \) and for all large values of \( r \),

\[
T(r, f) \leq (\rho_f^{**} + \varepsilon) \log r
\]

i.e.,

\[
\log T(r, f) \leq \log^{[2]} r + O(1)
\]

i.e.,

\[
\frac{\log T(r, f)}{\log^{[2]} r} \leq \frac{\log^{[2]} r + O(1)}{\log^{[2]} r}
\]

i.e.,

\[
\limsup_{r \to \infty} \frac{\log T(r, f)}{\log^{[2]} r} \leq 1 \tag{1}
\]

and

\[
\liminf_{r \to \infty} \frac{\log T(r, f)}{\log^{[2]} r} \leq 1. \tag{2}
\]
Again for arbitrary positive $\varepsilon$ and for all large values of $r$

\[
T (r, f) \geq (\lambda_f^{**} - \varepsilon) \log r
\]

i.e., \[
\log T (r, f) \geq \log^2 r + O (1)
\]

i.e., \[
\frac{\log T (r, f)}{\log^2 r} \geq \frac{\log^2 r + O (1)}{\log^2 r}
\]

i.e., \[
\limsup_{r \to \infty} \frac{\log T (r, f)}{\log^2 r} \geq 1
\]

(3)

and

\[
\liminf_{r \to \infty} \frac{\log T (r, f)}{\log^2 r} \geq 1.
\]

(4)

Therefore from (1) and (3) it follows that

\[
\rho_f^* = \limsup_{r \to \infty} \frac{\log T (r, f)}{\log^2 r} = 1.
\]

and from (2) and (4) we obtain that

\[
\lambda_f^* = \liminf_{r \to \infty} \frac{\log T (r, f)}{\log^2 r} = 1.
\]

Thus the theorem follows. ■

**Remark 1** If $f$ be any entire function of order zero. Then one can easily verify that (i) $\rho_f^* = 1$ and (ii) $\lambda_f^* = 1$.

**Example 1** Taking $f = z^2$ it can be easily verified that

\[
\rho_f = 0, \quad \rho_f^* = 1 \quad \text{and} \quad \rho_f^{**} = 2.
\]

Similarly if $g = z^3$, then

\[
\rho_g = 0, \quad \rho_g^* = 1 \quad \text{and} \quad \rho_g^{**} = 3.
\]

**Corollary 1** Under the conditions of Theorem 1,

\[
\lim_{r \to \infty} \frac{\log T (r, f)}{\log^2 r} = 1.
\]

If $f$ be an entire function of order zero then

\[
\lim_{r \to \infty} \frac{\log^2 M (r, f)}{\log^2 r} = 1.
\]
**Theorem 2** Let \( f \) and \( g \) be two entire functions such that \( \rho_f = 0 \) and \( \lambda_g < \infty \). Also let \( g \) be transcendental entire. Then \( \rho_{f \circ g} = \rho_g \).

**Proof.** In view of Lemma 1 and Theorem 1 we get that
\[
\rho_{f \circ g} = \limsup_{r \to \infty} \frac{\log [M(r, f \circ g)]}{\log r} \\
\leq \limsup_{r \to \infty} \frac{\log [M(r, g), f]}{\log [M(r, g)]} \limsup_{r \to \infty} \frac{\log [M(r, g)]}{\log r} \\
= \rho_f^* \rho_g = 1. \rho_g = \rho_g.
\]
(5)

Also from Lemma 2 and Theorem 1 it follows that
\[
\rho_{f \circ g} = \limsup_{r \to \infty} \frac{\log [M(r^{1+\delta}, f \circ g)]}{\log r^{1+\delta}} \\
\geq \liminf_{r \to \infty} \frac{\log [M(r, g), f]}{\log [M(r, g)]} \limsup_{r \to \infty} \frac{\log [M(r, g)]}{\log r} \\
= \lambda_f^* \rho_g = 1. \rho_g = \rho_g.
\]
(6)

Now combining (5) and (6) we obtain that
\[
\rho_{f \circ g} = \rho_g.
\]

This completes the proof. ■

**Theorem 3** Let \( f \) and \( g \) be two entire functions such that \( \rho_f = 0 \) and \( \lambda_g < \infty \). Also let \( g \) be transcendental entire. Then \( \frac{1}{3.4 \rho_g} \lambda_f^* \sigma_g \leq \sigma_{f \circ g} \leq \rho_f^* \sigma_g \).

**Proof.** In view of Lemma 1 and Theorem 2 we get that
\[
\sigma_{f \circ g} = \limsup_{r \to \infty} \frac{\log [M(r, f \circ g)]}{r^{\rho_{f \circ g}}} \\
\leq \limsup_{r \to \infty} \frac{\log [M(r, g), f]}{\log [M(r, g)]} \limsup_{r \to \infty} \frac{\log [M(r, g)]}{r^{\rho_g}} \\
= \rho_f^* \sigma_g.
\]

Again for all large values of \( r \),
\[
\log^+ M(r, f \circ g) \geq T(r, f \circ g) \\
\geq \frac{1}{3} \log M \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) + \circ (1), f \right) \quad \text{cf. [5] and [10]}
\]
Results of growth properties

\[ \log M(r, f \circ g) \geq \frac{1}{3} (\lambda_f^* - \varepsilon) \log \left\{ \frac{1}{8} M \left( \frac{r}{4}, g \right) + o(1) \right\} \]

\[ \log M(r, f \circ g) \geq \frac{1}{9} \lambda_f^* \log \left( \frac{r}{4}, g \right) \]

\[ \log M(r, f \circ g) \geq \frac{1}{3} (\lambda_f^* - \varepsilon) \log M \left( \frac{r}{4}, g \right) + O(1). \]

Therefore in view of Theorem 2 we get from above for all sufficiently large values of \( r \),

\[ \frac{\log M(r, f \circ g)}{r^{\rho_{f \circ g}}} \geq \frac{1}{3} (\lambda_f^* - \varepsilon) \frac{\log M \left( \frac{r}{4}, g \right) + O(1)}{r^{\rho_{f \circ g}}} \]

\[ \log M(r, f \circ g) \geq \frac{1}{3} (\lambda_f^* - \varepsilon) \frac{\log M \left( \frac{r}{4}, g \right)}{r^{\rho_{f \circ g}}} \frac{1}{4^{\rho_{f \circ g}}}. \]

So from above it follows that

\[ \limsup_{r \to \infty} \frac{\log M(r, f \circ g)}{r^{\rho_{f \circ g}}} \geq \frac{1}{3} (\lambda_f^* - \varepsilon) \limsup_{r \to \infty} \frac{\log M \left( \frac{r}{4}, g \right)}{r^{\rho_{f \circ g}}} \]

\[ \lim_{r \to \infty} \frac{\log M(r, g)}{r^{1+\delta}} \leq \frac{1}{3} (\lambda_f^* - \varepsilon) \sigma_g. \]

As \( \varepsilon (> 0) \) is arbitrary, the theorem is proved. \( \blacksquare \)

**Theorem 4** Let \( f \) be entire and \( g \) be transcendental entire with \( \lambda_g < \infty \). Also let \( \rho_{f \circ g} = 0 \), then

\[ \rho_f^* \lambda_g^* \leq \rho_{f \circ g}^* \leq \rho_f^* \rho_g^*. \]

**Proof.** By Lemma 2

\[ \rho_{f \circ g}^* = \limsup_{r \to \infty} \frac{\log M(r^{1+\delta}, f \circ g)}{\log r^{1+\delta}} \]

\[ \geq \limsup_{r \to \infty} \frac{\log M(M(r, g), f)}{\log M(r, g)} \liminf_{r \to \infty} \frac{\log M(r, g)}{\log r} \]

\[ = \rho_f^* \lambda_g^*. \]

Again by Lemma 1

\[ \rho_{f \circ g}^* = \limsup_{r \to \infty} \frac{\log M(r, f \circ g)}{\log r} \]

\[ \leq \limsup_{r \to \infty} \frac{\log M(M(r, g), f)}{\log M(r, g)} \limsup_{r \to \infty} \frac{\log M(r, g)}{\log r} \]

\[ = \rho_f^* \rho_g^*. \]

Thus the theorem is proved. \( \blacksquare \)
Theorem 5 Let $f$ be meromorphic and $g$ be transcendental entire such that $\rho_{fg} = 0$. Also let $0 < \lambda_{fg}^* \leq \rho_{fg}^* < \infty$ and $0 < \lambda_f^* \leq \rho_f^* < \infty$. Then for any positive number $A$

\[
\frac{\lambda_{fg}^*}{A\rho_f^*} \leq \liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{\lambda_{fg}^*}{A\lambda_f^*} \leq \limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{\rho_{fg}^*}{A\lambda_f^*}.
\]

Proof. Since $\rho_{fg} = 0 < \infty$ by Lemma 7, $\rho_f = 0$. Now from the definition of $\rho_f^*$ and $\lambda_f^*$ we have for arbitrary positive $\varepsilon$ and for all large values of $r$,

\[
T(r, f \circ g) \geq (\lambda_{fg}^* - \varepsilon) \log r
\] (7)

and

\[
T(r^A, f) \leq A \left( \rho_f^* + \varepsilon \right) \log r.
\] (8)

Now from (7) and (8) it follows for all large values of $r$,

\[
\frac{T(r, f \circ g)}{T(r^A, f)} \geq \frac{\lambda_{fg}^* - \varepsilon}{A \left( \rho_f^* + \varepsilon \right)}.
\]

As $\varepsilon (> 0)$ is arbitrary, we obtain that

\[
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \geq \frac{\lambda_{fg}^*}{A\rho_f^*}.
\] (9)

Again for a sequence of values of $r$ tending to infinity,

\[
T(r, f \circ g) \leq (\lambda_{fg}^* + \varepsilon) \log r
\] (10)

and for all large values of $r$,

\[
T(r^A, f) \geq A \left( \lambda_f^* - \varepsilon \right) \log r.
\] (11)

Combining (10) and (11) we get for a sequence of values of $r$ tending to infinity,

\[
\frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{(\lambda_{fg}^* + \varepsilon)}{A \left( \lambda_f^* - \varepsilon \right)}.
\]

Since $\varepsilon (> 0)$ is arbitrary it follows that

\[
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{\lambda_{fg}^*}{A\lambda_f^*}.
\] (12)
Results of growth properties

Also for a sequence of values of $r$ tending to infinity,

$$T(r^A, f) \leq A(\lambda^*_f + \varepsilon) \log r. \quad (13)$$

Now from (7) and (13) we obtain for a sequence of values of $r$ tending to infinity,

$$\frac{T(r, f \circ g)}{T(r^A, f)} \geq \frac{(\lambda^*_{f \circ g} - \varepsilon)}{A(\lambda^*_f + \varepsilon)}.$$  

As $\varepsilon (\geq 0)$ is arbitrary, we get from above that

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \geq \frac{\lambda^*_{f \circ g}}{A\lambda^*_f}. \quad (14)$$

Also for all large values of $r$,

$$T(r, f \circ g) \leq (\rho^*_{f \circ g} + \varepsilon) \log r. \quad (15)$$

So from (11) and (15) it follows for all large values of $r$,

$$\frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{(\rho^*_{f \circ g} + \varepsilon)}{A(\lambda^*_f - \varepsilon)}.$$  

Since $\varepsilon (\geq 0)$ is arbitrary we obtain that

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{\rho^*_{f \circ g}}{A\lambda^*_f}. \quad (16)$$

Thus the theorem follows from (9), (12), (14) and (16).  ■

In view of Lemma 8, the following theorem can be proved in the line of Theorem 5 and so the proof is omitted.

**Theorem 6** Let $f$ be meromorphic and $g$ be transcendental entire such that $\rho_{f \circ g} = 0$. Also let $0 < \lambda^*_f \leq \rho^*_{f \circ g} < \infty$, $\rho_f < \infty$ and $0 < \lambda^*_g \leq \rho^*_{g} < \infty$. Then for any positive number $A$

$$\frac{\lambda^*_{f \circ g}}{A\rho^*_g} \leq \liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, g)} \leq \frac{\lambda^*_{f \circ g}}{A\lambda^*_g} \leq \limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, g)} \leq \frac{\rho^*_{f \circ g}}{A\lambda^*_g}.$$ 

**Theorem 7** Let $f$ be meromorphic and $g$ be entire such that $\rho_{f \circ g} = 0$. Also let $0 < \lambda^*_f \leq \rho^*_{f \circ g} < \infty$ and $0 < \rho^*_g < \infty$. Then for any positive number $A$

$$\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \leq \frac{\rho^*_{f \circ g}}{A\rho^*_f} \leq \limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)}.$$
Proof. In view of Lemma 7, \( \rho_{f \circ g} = 0 \) implies that \( \rho_f = 0 \).
From the definition of \( \rho_f^{**} \) we get for a sequence of values of \( r \) tending to infinity,

\[
T (r^A, f) \geq A \left( \rho_f^{**} - \varepsilon \right) \log r.
\]  
(17)

Now from (15) and (17) it follows for a sequence of values of \( r \) tending to infinity,

\[
\frac{T (r, f \circ g)}{T (r^A, f)} \leq \frac{\rho_{f \circ g}^{**} + \varepsilon}{A \left( \rho_f^{**} - \varepsilon \right)}.
\]

As \( \varepsilon (> 0) \) is arbitrary we obtain that

\[
\liminf_{r \to \infty} \frac{T (r, f \circ g)}{T (r^A, f)} \leq \frac{\rho_{f \circ g}^{**}}{A \rho_f^{**}}.
\]  
(18)

Again for a sequence of values of \( r \) tending to infinity,

\[
T (r, f \circ g) \geq \left( \rho_{f \circ g}^{**} - \varepsilon \right) \log r.
\]  
(19)

So combining (8) and (19) we get for a sequence of values of \( r \) tending to infinity,

\[
\frac{T (r, f \circ g)}{T (r^A, f)} \geq \frac{\rho_{f \circ g}^{**} - \varepsilon}{A \left( \rho_f^{**} + \varepsilon \right)}.
\]

Since \( \varepsilon (> 0) \) is arbitrary it follows that

\[
\limsup_{r \to \infty} \frac{T (r, f \circ g)}{T (r^A, f)} \geq \frac{\rho_{f \circ g}^{**}}{A \rho_f^{**}}.
\]  
(20)

Thus the theorem follows from (18) and (20).

In view of Lemma 8, the following theorem can be carried out in the line of Theorem 7 and therefore we omit the proof.

Theorem 8 Let \( f \) be meromorphic and \( g \) be entire such that \( \rho_{f \circ g} = 0 \). Also let \( 0 < \lambda_{f \circ g}^{**} \leq \rho_{f \circ g}^{**} < \infty, \rho_f < \infty \) and \( 0 < \rho_g^{**} < \infty \). Then for any positive number \( A \)

\[
\liminf_{r \to \infty} \frac{T (r, f \circ g)}{T (r^A, g)} \leq \frac{\rho_{f \circ g}^{**}}{A \rho_g^{**}} \leq \limsup_{r \to \infty} \frac{T (r, f \circ g)}{T (r^A, g)}.
\]

The following theorem is a natural consequence of Theorem 5 and Theorem 7.
Theorem 9 Let $f$ be meromorphic and $g$ be entire such that $\rho_{fog} = 0$. Also let $0 < \lambda_{fog}^* \leq \rho_{fog}^* < \infty$ and $0 < \lambda_g^* \leq \rho_g^* < \infty$. Then for any positive number $A$,

$$
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)} \leq \min \left\{ \frac{\lambda_{fog}^*}{A \lambda_f^*}, \frac{\rho_{fog}^*}{A \rho_f^*} \right\} \\
\leq \max \left\{ \frac{\lambda_{fog}^*}{A \lambda_f^*}, \frac{\rho_{fog}^*}{A \rho_f^*} \right\} \leq \limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, f)}.
$$

The proof is omitted.

Analogously one may state the following theorem without proof.

Theorem 10 Let $f$ be meromorphic and $g$ be entire such that $\rho_{fog} = 0$. Also let $0 < \lambda_{fog}^* \leq \rho_{fog}^* < \infty$, $\rho_f < \infty$ and $0 < \lambda_g^* \leq \rho_g^* < \infty$. Then for any positive number $A$,

$$
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, g)} \leq \min \left\{ \frac{\lambda_{fog}^*}{A \lambda_g^*}, \frac{\rho_{fog}^*}{A \rho_g^*} \right\} \leq \max \left\{ \frac{\lambda_{fog}^*}{A \lambda_g^*}, \frac{\rho_{fog}^*}{A \rho_g^*} \right\} \leq \limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, g)}.
$$

Theorem 11 Let $f$ be a meromorphic function of order zero and $g$ be entire such that $\rho_g$ is finite. Then

$$
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \leq (1 + o(1)) \rho_f^* 2^{\rho_g^*}.
$$

Proof. If $\rho_f^* = \infty$, then the result is obvious. So we suppose that $\rho_f^* < \infty$. Since $T(r, g) \leq \log^+ M(r, g)$, we obtain by Lemma 3 for $\varepsilon (> 0)$ and for all large values of $r$,

$$
T(r, f \circ g) \leq (1 + o(1)) (\rho_f^* + \varepsilon) \log M(r, g)
$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \leq (1 + o(1)) \rho_f^* \liminf_{r \to \infty} \frac{\log M(r, g)}{T(r, g)}.
$$

Since $\limsup_{r \to \infty} \frac{T(r, g)}{r^\rho_g(r)} = 1$, for given $\varepsilon (0 < \varepsilon < 1)$ we get for all large values of $r$,

$$
T(r, g) < (1 + \varepsilon) r^\rho_g(r)
$$

(22)
and for a sequence of values of \( r \) tending to infinity

\[
T(r, g) > (1 - \varepsilon) r^{\rho_g(r)}.
\]  

Since \( \log M(r, g) \leq 3T(2r, g) \), for a sequence of values of \( r \) tending to infinity we get for any \( \delta > 0 \)

\[
\frac{\log M(r, g)}{T(r, g)} \leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} \frac{(2r)^{\rho_g + \delta}}{(2r)^{\rho_g} - \rho_g(2r)} \frac{1}{r^{\rho_g(r)}}
\]

\[
\leq \frac{3(1 + \varepsilon)}{(1 - \varepsilon)} 2^{\rho_g + \delta}.
\]

because \( r^{\rho_g + \delta - \rho_g(r)} \) is ultimately an increasing function of \( r \). Since \( \varepsilon > 0 \) and \( \delta > 0 \) are arbitrary, we obtain that

\[
\liminf_{r \to \infty} \frac{\log M(r, g)}{T(r, g)} \leq 3.2^{\rho_g}.
\]  

Thus from (21) and (24) it follows that

\[
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \leq (1 + o(1)) 3.\rho_f^{**} 2^{\rho_g}.
\]

Theorem 12 Let \( f \) be a meromorphic function of order zero and \( g \) be entire with \( \lambda_g < \infty \). Then

\[
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \leq (1 + o(1)) 3.\rho_f^{**} 2^{\rho_g}.
\]

Theorem 13 Let \( f \) and \( g \) be two non constant entire functions such that \( f \) is of lower order zero and \( \lambda_f^{**} \) and \( \lambda_g \) are finite. Then

\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \geq (1 + o(1)) \frac{1}{3} \frac{\lambda_f^{**}}{4\lambda_g}.
\]

Proof. If \( \lambda_f^{**} = 0 \) then the result is obvious. So we suppose that \( \lambda_f^{**} > 0 \). For all values of \( r \) we know that

\[
T(r, f \circ g) \geq \frac{1}{3} \log M \left\{ \frac{1}{8} M\left(\frac{r}{4}, g\right) + o(1), f \right\} \{c.f. \[10]\}\]
For \( 0 < \varepsilon < \min \{ \lambda_f^{**}, 1 \} \) we get for all large values of \( r \),

\[
T(r, f \circ g) \geq \frac{1}{3} (\lambda_f^{**} - \varepsilon) \log \left\{ \frac{1}{8} M \left( \frac{r}{4}, g \right) + o(1) \right\}
\]

i.e.,

\[
T(r, f \circ g) \geq \frac{1}{3} (\lambda_f^{**} - \varepsilon) \log \left\{ \frac{1}{9} M \left( \frac{r}{4}, g \right) \right\}
\]

i.e.,

\[
T(r, f \circ g) \geq \frac{1}{3} (\lambda_f^{**} - \varepsilon) \log \left\{ \frac{1}{3} (\lambda_f^{**} - \varepsilon) \log \frac{1}{9} \right\}
\]

i.e.,

\[
T(r, f \circ g) \geq \frac{1}{3} (\lambda_f^{**} - \varepsilon) T \left( \frac{r}{4}, g \right) + O(1).
\] (25)

Since \( \lim \inf_{r \to \infty} \frac{T(r, g)}{r^{\lambda_g(r)}} = 1 \) for given \( \varepsilon (>0) \) we get for all large values of \( r \)

\[
T(r, g) > (1 - \varepsilon) r^{\lambda_g(r)}.
\] (26)

and for a sequence of values of \( r \) tending to infinity

\[
T(r, g) < (1 + \varepsilon) r^{\lambda_g(r)}.
\] (27)

From (25) and (26) we get for \( \delta (>0) \) and for all large values of \( r \)

\[
T(r, f \circ g) \geq \frac{1}{3} (\lambda_f^{**} - \varepsilon) (1 - \varepsilon) (1 + o(1)) \frac{r^{\lambda_g + \delta}}{(\frac{r}{4})^{\lambda_g + \delta - \lambda_g(\frac{r}{4})}}.
\]

Since \( r^{\lambda_g + \delta - \lambda_g(r)} \) is ultimately an increasing function of \( r \) it follows for all large values of \( r \) that

\[
T(r, f \circ g) \geq \frac{1}{3} (\lambda_f^{**} - \varepsilon) (1 - \varepsilon) (1 + o(1)) \frac{r^{\lambda_g(r)}}{4^{\lambda_g + \delta}}.
\] (28)

So by (27) and (28) we get for a sequence of values of \( r \) tending to infinity

\[
T(r, f \circ g) \geq \frac{1}{3} (\lambda_f^{**} - \varepsilon) \frac{(1 - \varepsilon)}{(1 + \varepsilon)} (1 + o(1)) \frac{T(r, g)}{4^{\lambda_g + \delta}}.
\]

Since \( \varepsilon (>0) \) and \( \delta (>0) \) are arbitrary it follows from above that

\[
\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \geq (1 + o(1)) \frac{1}{3} \frac{\lambda_f^{**}}{4^{\lambda_g}}.
\]

Thus the theorem is proved. \( \blacksquare \)
**Theorem 14** Let $f$ and $g$ be two non constant entire functions such that $\rho_f^*$ and $\lambda_g$ are finite. Also suppose that there exist entire functions $a_i$ ($i = 1, 2, \ldots, n; n \leq \infty$) satisfying

(i) $T(r, a_i) = o\{T(r, g)\}$ as $r \to \infty$ for $i = 1, 2, \ldots, n$

(ii) $\sum_{i=1}^{n} \delta(a_i, g) = 1$.

Then

$$\frac{\pi \lambda_f^{**}}{3.4^{\lambda_g}} \leq \limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \leq \pi \rho_f^{**}.$$

**Proof.** For any two entire functions $f$ and $g$, the following two inequalities are well known,

$$T(r, f) \leq \log^+ M(r, f) \leq 3T(2r, f). \ {cf.}[5]$$

and,

$$\log M(r, f \circ g) \leq \log M(M(r, g), f). \ {cf.}[2]$$

For $\varepsilon > 0$ we get from (29) and (30) for all large values of $r$,

$$T(r, f \circ g) \leq \log M(M(r, g), f)$$

i.e.,

$$T(r, f \circ g) \leq \left(\rho_f^{**} + \varepsilon\right) \log M(r, g)$$

i.e.,

$$\frac{T(r, f \circ g)}{T(r, g)} \leq \left(\rho_f^{**} + \varepsilon\right) \frac{\log M(r, g)}{T(r, g)}.$$

Hence we get from above that

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \leq \left(\rho_f^{**} + \varepsilon\right) \limsup_{r \to \infty} \frac{\log M(r, g)}{T(r, g)}.$$

Since $\varepsilon > 0$ is arbitrary, it follows from Lemma 4 that

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \leq \pi \rho_f^{**}. \quad (31)$$

Now suppose that $0 < \varepsilon < \min\{\lambda_f^{**}, 1\}$ we get from (25) for all large values of $r$ that

$$\frac{T(r, f \circ g)}{T(r, g)} \geq \frac{1}{3} \left(\lambda_f^{**} - \varepsilon\right) \frac{\log M(\frac{r}{\lambda}, g)}{T(\frac{r}{\lambda}, g)} \frac{T(\frac{r}{\lambda}, g)}{T(r, g)} + O(1). \quad (32)$$

From (26) and (27) and in the line of Lemma 6 we get for a sequence of values of $r$ tending to infinity and for $\delta > 0$

$$\frac{T(\frac{r}{\lambda}, g)}{T(r, g)} \geq \frac{1 - \varepsilon}{1 + \varepsilon} \frac{(\frac{r}{\lambda})^{\lambda_g + \delta}}{1 + \varepsilon (\frac{r}{\lambda})^{\lambda_g + \delta}} \frac{1}{T^\lambda(\frac{r}{\lambda})} \geq \frac{1 - \varepsilon}{1 + \varepsilon} \frac{1}{4^{\lambda_g + \delta}}.$$
Since $\varepsilon (> 0)$ and $\delta (> 0)$ are arbitrary we get from Lemma 4, (32) and above that

$$\limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} \geq \frac{\pi \lambda_f^{**}}{3.4^h}.$$  (33)

Thus the theorem follows from (31) and (33).

References


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