Some Observations about Infinity-Harmonic Functions on $\mathbb{R}^n$

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Abstract

In this note we present some observations about infinity-harmonic functions on $\mathbb{R}^n$, $n \geq 2$. We also discuss conditions under which an infinity-harmonic function on $\mathbb{R}^n$ is affine.

1 Introduction

In this work we present an observation related to infinity-harmonic functions on $\mathbb{R}^n$. Such functions have been found to be useful in viscoelasticity, the Monge-Kantorovich mass transfer problem and in the problem of finding functions with minimal Lipschitz extensions. Our work, however, will be limited to infinity-harmonic functions, on $\mathbb{R}^n$, with bounded gradients. A conjecture, attributed to Crandall and Evans, states that such functions are affine. This has been proven to be true in $n = 2$ [9]. We direct the reader to the works in [7, 8, 9] where local regularity issues are addressed and a proof of $C^{1,\alpha}$ in $n = 2$ is presented [8]. While these works are quite deep and provide an approach to the general regularity issue, the matter of what happens in $n \geq 3$ is as yet unclear. We introduce some notations for our discussion. A function $u$ is infinity-harmonic in a domain $\Omega \subset \mathbb{R}^n$, if it is a viscosity solution to

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \quad \text{in } \Omega.$$ 

Let $o$ denote the origin in $\mathbb{R}^n$, and $B_r(x)$ be the ball of radius $r$, centered at $x$. Define $M_r(x) = \sup_{B_r(x)} u$ and $m_r(x) = \inf_{B_r(x)} u$. It is well known that

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$M_r(x)$, as a function of $r$ is convex, while $m_r(x)$ is concave. Moreover,

$$\lim_{r \downarrow 0} \frac{M_r(x) - u(x)}{r} = \lim_{r \downarrow 0} \frac{u(x) - m_r(x)}{r} = \Lambda(x) < \infty,$$

where the quantity $\Lambda(x)$ will equal the modulus of the gradient $|Du(x)|$, if $u$ were differentiable at $x$. For more detailed discussions, see [1, 2, 4, 6, 7]. The function $\Lambda$ obeys a maximum principle, namely, if $\Lambda_r(x) = \sup_{y \in B_r(x)} \Lambda(y)$, then $\Lambda_r(x) = \sup_{z \in \partial B_r(x)} \Lambda(z)$ [5]. We will assume in this work that $u$, infinity-harmonic on $\mathbb{R}^n$, is such that $\sup_{x \in \mathbb{R}^n} \Lambda(x) = 1$. Our effort in this note is to show that under some additional conditions the conjecture is true and $u$ is affine. We elaborate on this in the next section. For a related work, see [3].

2 Some observations

Let $x \in \mathbb{R}^n$, and $p_r, q_r$ denote points on $\partial B_r(x)$, where $u$ attains its maximum and minimum respectively. Set $\omega_r = (p_r - x)/r$; we now state some simple conclusions [4, 5, 6], see in particular Theorem 2 in [4]. By the convexity of $M_r$ and the monotonicity of $\Lambda$, we see that

$$\Lambda(x) \leq \frac{M_r(x) - u(x)}{r} \leq \frac{M_r(x) - u(t\omega_r + x)}{r} \leq \Lambda(p_r) \leq 1. \quad (1)$$

Moreover, the following properties also hold. Let $\eta \in S^{n-1}$, then for $0 < t < s < r$,

$$\frac{M_r(x) - u(x + s\eta)}{r - s} \geq \frac{M_r(x) - u(x + t\eta)}{r - t}, \quad (M_r(x) - u(x + s\eta))(r - s) \leq (M_r(x) - u(x + t\eta))(r - t). \quad (2)$$

Analogous statements hold for $m_r(x)$. We start with an elementary observation. Also compare with Lemma 3.1(a) in [5]. Note that $M'_r(x) = dM_r(x)/dr$; since $M$ is convex, $M'_r(x)$ exists for almost every $r$.

**Lemma 1** Let $u$ be infinity-harmonic in $\mathbb{R}^n$ and $\sup_{x \in \mathbb{R}^n} \Lambda(x) = 1$. Then for every $x \in \mathbb{R}^n$, $\lim_{r \uparrow \infty} M_r(x)/r = \lim_{r \uparrow \infty} M'_r(x) = 1$. An analogous statement holds for $m_r(x)$.

**Proof.** For a cleaner exposition, we set $M(r) = M_r(x)$. By convexity, $(M(r) - u(x))/r$ and $M'(r)$ both increase with $r$. Note that $M(r)$ is differentiable for almost every $r$ and $M'(r-)$ exists. The statement made for $M'(r)$ is to be understood in the sense of almost everywhere, although the
claim holds for both $M'(r^-)$ and $M'(r^+)$. Moreover, if $p_r \in \partial B_r(x)$ is a point of maximum then,

$$\frac{M(r) - u(x)}{r} \leq M'(r^-) \leq \Lambda(p_r) \leq M'(r^+) \leq 1, \quad \forall r > 0. \quad (3)$$

See Theorem 2 in [4]. Let $\varepsilon > 0$ and $y \in \mathbb{R}^n$ be such that $1 - \varepsilon \leq \Lambda(y) \leq 1$. Let $\rho > 0$ and take $r = |x - y| + \rho$. By applying (2) in $B_r(x)$, $(M(r) - u(y))\rho \leq (M(r) - u(x))r$; thus (1) implies that

$$\Lambda(y) \leq \frac{M_\rho(y) - u(y)}{\rho} \leq \frac{M(r) - u(y)}{\rho} \leq \frac{M(r) - u(x)}{r} \left(\frac{r}{r - |x - y|}\right)^2$$

Letting $r \to \infty$, we obtain the statement of the Lemma. □

We now make one of our main observations.

**Lemma 2** (Affine property) Let $u$ be infinity-harmonic function on $\mathbb{R}^n$ and $\omega \in S^{n-1}$ be such that (i) $u(o) = 0$, and (ii) $u(r\omega) = M(r) = r$. Then (a) $m(r) = u(-r\omega) = -r$, and (b) $u$ is affine.

**Proof.** Without any loss of generality, we take $\omega = e_n$, the unit vector along the positive $x_n$-axis. Our effort will then be to show that $u(x) = x_n$, $x \in \mathbb{R}^n$. Our proof will use the properties stated in (2). By our hypothesis, it is clear that $\Lambda(o) = 1$, $u$ is differentiable at $o$ and $Du(o) = e_n$, see [4, 7]. Let $r > 0$, it is clear that $m(r) < 0$. By (1) and Lemma 1, $1 = \Lambda(o) \leq |m(r)|/r$, implying that $m(r) \leq -r$. Let $\alpha > 1$; applying (2) in the ball $B_{\alpha r}(o)$, we see that

$$(M(\alpha r) - u(o))(\alpha r) \geq (M(\alpha r) - m(r))(\alpha - 1)r \quad \text{and} \quad |m(r)| \leq \frac{\alpha r}{\alpha - 1}$$

Letting $\alpha \to \infty$ and using the analogue of (1) for $m(r)$, we see that $m(r) = -r$. Also $u(-r e_n) \geq m(r) = -r$. This clearly implies that $m'(r) = -1$. Let $q_r \in \partial B_r(o)$ such that $u(q_r) = m(r)$, then

$$1 = \Lambda(o) \leq \frac{u(t \nu_r) - m(r)}{r - t} \leq \Lambda(q_r) = |m'(r)| = 1,$$

where $\nu_r = q_r/r$. Thus $u(t \nu_r) = -t$, $0 \leq t \leq r$. Since $u$ is differentiable at $o$, we see that $\lim_{t \to 0} u(t \nu_r)/t = \langle e_n, \nu_r \rangle = -1$. Clearly, $\nu_r = -e_n$ and $u(-r e_n) = m(r) = -r$, $r \geq 0$. This proves part (a) of the Lemma.

We now show part (b). Let $H$ be the $n - 1$ hyperplane, with normal $e_n$ and passing through $o$. Select $y \in H$, $y \neq o$. Let $r > 0$ and set $d = \sqrt{r^2 + |y|^2}$. We take $\alpha > 1$. Since $B_{ad}(re_n) \subset B_{ad + r}(o)$, it follows by hypothesis (ii) that
follows from Lemma 1 that $M(y)$ points $t > 0$. Thus Lemma 2 holds without requiring that $\sup x \in B(r, \rho)$.

We next discuss how this applies to some simple situations related to the conjecture and a possible motivation. By the upper semi-continuity property and the maximum principle for $\Lambda(x)$ \cite{5, 7}, if $\sup x \Lambda(x) = 1$ then either (i) there is a $z$ such that $\Lambda(z) = 1$, or (ii) there is a sequence $z_k$ with $|z_k| \to \infty$ such that $\Lambda(z_k) \uparrow 1$. We address (i) first. The second possibility is the harder of the two and at this time it is not clear how to handle it.

Lemma 3 Let $u$ be infinity-harmonic on $\mathbb{R}^n$, and suppose that $\sup x \Lambda(x) = 1$. The following hold.
(i) If there is a point \( z \) such that \( \Lambda(z) = 1 \), then \( u \) is affine.

(ii) Assume that \( u(0) = 0 \). Let \( r > 0 \), set \( M(r) = \sup_{B_r(0)} u \), and \( E(r) = M'(r) - M(r)/r \), for almost every \( r \). If \( rE(r) \to 0 \) as \( r \to \infty \) then \( u \) is affine.

**Proof.** We present the proof of (i). Let \( z \) be such that \( \Lambda(z) = 1 \); for notational ease, set \( M(r) = M_r(z) \). By (1), (3) and Lemma 1,

\[
1 = \Lambda(z) \leq \frac{M(r) - u(z)}{r} \leq M'(r-) \leq M'(r+) \leq 1, \quad \forall \ r > 0.
\]

Clearly, \( M_r(z) = r + u(z), \ \forall \ r > 0 \). Moreover, using (2), we know that if \( \eta_r \in S^{n-1} \) is such that \( u(z + r\eta_r) = M_r(z) \) then

\[
\Lambda(z) \leq \frac{M_r(z) - u(z)}{r} \leq \frac{M_r(z) - u(z + t\eta_r)}{r - t} \leq \Lambda(z + r\eta_r) = 1, \quad 0 \leq t < r. \quad (4)
\]

This implies that \( u \) is differentiable at \( z \), \( Du(z) = \eta_r \) and \( u(z + t\eta_r) = t + u(z) = M_r(z) \). Fix \( r > 0 \) and set \( \eta = \eta_r \). Let \( L \) be the straight line \( L = \{ z + s\eta : -\infty < s < \infty \} \). We will show that \( u \) is linear on \( L \) and \( u(z + s\eta) = u(z) + s = M_s(z), \ \forall \ s > 0 \), also see Lemma 3.4 in [5]. In order to prove our claim, fix \( s > 0 \) and use (4) with \( r \) replaced by \( s \). Since \( u \) is differentiable at \( z \), it follows that \( \eta = \eta_s \) and the conclusion follows. Clearly, the function \( v_z(x) = u(x) - u(z) \) satisfies the hypothesis in Lemma 2 with \( z \) in place of \( o \) and it follows that \( u \) is affine in \( \mathbb{R}^n \).

We make the following observation before proving part (ii) of the Lemma. Suppose that \( x \in \mathbb{R}^n \) and \( B_r(x) \) are such that either \( (M_r(x) - u(x))/r = \Lambda(x) \) or \( (M_r(x) - u(x))/r = \Lambda(p_r) \), where \( p_r \in \partial B_r(x) \) and \( u(p_r) = M_r(x) \). In either case, it follows that \( \Lambda(x) = \Lambda(p_r) \) and \( u(x + t\omega_r) = u(x) + t\Lambda(x) \), where \( \omega_r = p_r/r \). Moreover, \( u \) is differentiable at \( x \) and \( Du(x) = \Lambda(x)\omega_r \). In other words \( u \) is linear along the segment \( xp_r \), see [5]. In the current situation, however, this equality is attained when \( |x| \to \infty \), see Lemma 1. Again from Lemma 1 (now working in \( B_r(o) \)), we know that if \( E(r) = M'(r) - M(r)/r \), then \( E(r) \to 0 \) as \( r \to \infty \). Let \( p_r \) be a point of maximum and \( \omega_r \) be the direction \( p_r/r \). Recalling that \( M'(r-) \leq \Lambda(p_r) \leq M'(r+) \), using \( E(r) \), convexity and (1), we have for \( 0 < t < r \),

\[
\Lambda(p_r) - E(r) \leq \frac{M(r)}{r} \leq \frac{M(r) - M(t)}{r - t} \leq \frac{M(r) - u(t\omega_r)}{r - t} \leq \Lambda(p_r), \quad r > 0.
\]

The above holds for almost every \( r \) and for every \( t < r \). Rearranging terms, we obtain the following inequalities:

\[
t\Lambda(p_r) + (M(r) - \Lambda(p_r)r) \leq u(t\omega_r) \leq M(t) \leq (M(r) - \Lambda(p_r)r) + E(r)(r - t) + t\Lambda(p_r) \quad (5)
\]
At this time we have been unable to prove that \( rE(r) \to 0 \) and it is also unclear if this is really needed. We now make our final observation along the lines of the works in \([4, 7]\).

**Lemma 4** Let \( u \) be infinity-harmonic in \( \mathbb{R}^n \), \( u(o) = 0 \), and suppose that \( \sup_x \Lambda(x) = 1 \). For \( r > 0 \), consider the ball \( B_r(o) \) and let \( \omega_r \in S^{n-1} \) be such that \( u(r\omega_r) = M(r) \).

(a) Suppose that \( r_k \uparrow \infty \) is such that \( \omega_k = \omega_{r_k} \to \omega \), for some \( \omega \in S^{n-1} \). Then \( \omega \) is also a limit direction at any \( x \in \mathbb{R}^n \), as \( r_k \uparrow \infty \). Moreover, if \( \eta \in S^{n-1} \) then, for any \( \theta \geq 1 \), \( \lim_{r_k \uparrow \infty} u(\theta r_k \eta)/r_k = \theta(\omega, \eta) \). This also holds at any \( x \in \mathbb{R}^n \).

(b) Let \( \omega \) and the sequence \( r_k \) be as in part (a). Define \( a(r) = \sup \{ b \in \mathbb{R} : b + \langle \omega, x \rangle \leq u(x), x \in \partial B_r(o) \} \). Then \( a(r) \) is decreasing in \( r \) and \( \lim_{r \to 0} a(r_k)/r_k = 0 \). Furthermore, if \( \omega \) is the only limit direction then \( \lim_{r \to 0} a(r)/r = 0 \). Analogous conclusions hold at \( x \neq o \).

**Proof.** We achieve the proof of (a) in five steps. By Lemma 1, for any \( \theta > 0 \), \( \lim_{k \to \infty} M(\theta r_k)/r_k = \theta \), which also holds for \( |m(\theta r_k)| \). The main statements we need for our proof are (1) \( \lim_{k \to \infty} u(\theta r_k \omega)/r_k = \theta, \theta > 0 \), and (2) if \( \eta_k \) is such that \( u(\omega_{r_k} \eta_k) = m(r_k) \) then \( \eta_k \to -\omega \). These will be shown in Steps (ii) and (iii), while the claim will be proven in Steps (iv) and (v). The idea is analogous to the one used in proving Theorem 3 in \([4]\) and relies on the use of the Harnack inequality. We also point out that \([7]\) contains a version proved by using a different method. Both the works deal with the case \( r_k \downarrow 0 \) in connection with differentiability at a point.

Step (i): Take \( \theta = 1 \), \( \alpha > 1 \), and let \( r_k \) and \( \omega_k \) be as in the statement of the lemma. Applying the Harnack inequality in \( B_{\alpha r}(o) \) (see \([1, 3]\)), we see that

\[
\frac{M(\alpha r_k) - u(r_k \omega_k)}{r_k} \geq \frac{M(\alpha r_k) - u(r_k \omega)}{r_k} e^{-|\omega_k - \omega|/(\alpha-1)}.
\]

Noting that \( u(r_k \omega_{r_k}) = M(r_k) \), selecting an appropriate subsequence of \( r_k \) and letting \( r_k \to \infty \), Lemma 1 yields \( \alpha - 1 \geq \alpha - \liminf_{k \to \infty} u(r_k \omega)/r_k \). Thus \( \liminf_{k \to \infty} u(r_k \omega)/r_k \geq 1 \). Arguing analogously, \( \limsup_{k \to \infty} u(r_k \omega)/r_k \leq 1 \). Thus \( \lim_{k \to \infty} u(r_k \omega)/r_k = 1 \).
Step (i): We now show that if \( \eta_k \in S^{n-1} \) is such that \( u(r_k \eta_k) = m(r_k) \) then \( \eta_k \to -\omega \). Let \( \eta \) be any limit point of \( \eta_k \). Applying the Harnack inequality in \( B_{ar_k}(o) \), \( \alpha > 1 \), it follows that

\[
\frac{M(\alpha r_k) - u(r_k \omega_k)}{r_k} \geq \frac{M(\alpha r_k) - u(r_k \eta_k)}{r_k} e^{-|\omega_k - \eta_k|/(\alpha - 1)}
\]

Letting \( r_k \to \infty \) (or a subsequence, if needed) and using Lemma 1, we find that \( e^{\omega - \eta} \geq \left( \frac{\alpha + 1}{\alpha - 1} \right)^{\alpha - 1} \to e^2 \), as \( \alpha \to \infty \). This implies that \( \eta = -\omega \). Following the ideas of Step (i), we may now show that \( u(-r_k \omega)/r_k \to -1 \) as \( r_k \to \infty \).

Step (ii): We now show that if \( \eta \) be any limit point of \( \eta_k \), then \( \eta \in B \). Letting \( \eta \to -\omega \), it follows that \( \eta \in B \). Letting \( r_k \to \infty \) (or a subsequence, if needed), using Lemma 1, we obtain that

\[
\frac{M(\alpha r_k) - u(\eta r_k \omega_k)}{r_k} \geq \frac{M(\alpha r_k) - u(-r_k \omega_k)}{r_k} e^{-|\eta \omega_k + \omega|/(\alpha - \theta)}
\]

Letting \( r_k \to \infty \) (or a subsequence, if needed) and using Lemma 1, we obtain that \( e^{|\eta \omega_k + \omega|} \geq \left( \frac{\alpha + 1}{\alpha - \theta} \right)^{\alpha - \theta} \to e^{\theta + 1} \), as \( \alpha \to \infty \). It is clear that \( \omega = \omega \). Using Step (i) (replacing \( r_k \) by \( \eta r_k \)), we may now conclude that \( \lim_{k \to \infty} u(\eta r_k \omega)/r_k = \theta, \theta > 1 \). It is not difficult to show that the statement holds for any \( \theta > 0 \). Using analogous arguments and Step 2, one can show that \( \lim_{r_k \to \infty} u(-\theta r_k \omega) = -\theta \).

Step (iii): Now take \( \theta > 1 \). Let \( \omega^\theta_k \in S^{n-1} \) be such that \( u(\theta r_k \omega^\theta_k) = m(\theta r_k) \). Take \( \alpha > \theta \) and let \( \omega^\theta \) be a limit point of \( \omega^\theta_k \). Applying the Harnack inequality in \( B_{ar_k}(o) \), we obtain that

\[
\frac{M(\alpha r_k) - u(\theta r_k \omega^\theta_k)}{r_k} \geq \frac{M(\alpha r_k) - u(-r_k \omega^\theta)}{r_k} e^{-|\theta \omega^\theta + \omega|/(\alpha - \theta)}
\]

Letting \( r_k \to \infty \) (or a subsequence, if needed) and using Lemma 1, we obtain that \( e^{|\theta \omega^\theta + \omega|} \geq \left( \frac{\alpha + 1}{\alpha - \theta} \right)^{\alpha - \theta} \to e^{\theta + 1} \), as \( \alpha \to \infty \). It is clear that \( \omega = \omega^\theta \). Using Step (i) (replacing \( r_k \) by \( \theta r_k \)), we may now conclude that \( \lim_{k \to \infty} u(\theta r_k \omega)/r_k = \theta, \theta > 1 \). It is not difficult to show that the statement holds for any \( \theta > 0 \). Using analogous arguments and Step 2, one can show that \( \lim_{r_k \to \infty} u(-\theta r_k \omega) = -\theta \).

Step (iv): Now let \( x \neq o \) and \( r_k \) be as in part (a) of the Lemma. Select \( e_k \in S^{n-1} \) such that \( u(x + r_k e_k) = M(x) \). We claim that \( \omega \) is also the limit of the sequence \( e_k \), as \( r_k \to \infty \). To see this we apply the Harnack inequality as follows. Let \( \alpha > \beta > 1 \). First we take \( r_k \) large so that \( x + r_k e_k \in B_{3r_k}(o) \). Let \( e_k \to e \) (selecting a subsequence if needed). Then by the Harnack inequality

\[
\frac{M(\alpha r_k) - M(x + r_k e_k)}{r_k} \geq \frac{M(\alpha r_k) - u(-\beta r_k \omega)}{r_k} \exp \left( -\frac{\beta \omega + e_k}{\alpha - \beta} + \frac{x}{(\alpha - \beta) r_k} \right)
\]

Letting \( k \to \infty \) (or a subsequence, if needed), using Lemma 1 and Step (ii), we see that

\[
e^{\beta \omega + e} \geq \left( \frac{\alpha + \beta}{\alpha - 1} \right)^{\alpha - \beta} \text{ implying } |\beta \omega + e| \geq (\beta + 1),
\]

where the second inequality follows by taking \( \alpha \to \infty \). Simplifying, it follows that \( e = \omega \).
Step (v). Let \( \nu \in S^{n-1} \) and let \( \alpha > \beta > 1 \). We apply the Harnack inequality in \( B_\alpha(o) \) as follows.

\[
\frac{M(\alpha r_k) - u(r_k \nu)}{r_k} \geq \frac{M(\alpha r_k) - u(-\beta r_k \omega)}{r_k} e^{-|\beta \omega + \nu|/(\alpha - \beta)}
\]

Setting \( \limsup_{r_k \to \infty} u(r_k \nu)/r_k = U \leq 1 \), using Step (iii) and an appropriate subsequence of \( r_k \), we find that

\[
e^{\alpha U} \geq \left( \frac{\alpha + \beta}{\alpha - U} \right)^{\alpha - \beta} \implies |\beta \omega + \nu| \geq (\beta + U).
\]

Simplifying and then letting \( \beta \to \infty \), we see that \( U \leq \langle \omega, \nu \rangle \). In analogous fashion, one can show that \( \liminf_{r_k \to \infty} u(r_k \nu)/r_k \geq \langle \omega, \nu \rangle \). We have thus shown part (a). Clearly by Step (iv), this also holds at any \( x \).

We now prove part (b) of the lemma. Let \( a(r) \) be as in the statement of the lemma. Clearly for a fixed \( r \), \( a(r) + \langle \omega, x \rangle \) is infinity-harmonic in \( B_r(o) \). Using comparison, \( a(r) + \langle \omega, x \rangle \leq u(x) \) in \( B_r(o) \). Since \( u(o) = 0 \), it follows that \( a(r) \leq 0 \). Moreover, there is an \( \eta_r \in S^{n-1} \) such that \( u(r \eta_r) = r \langle \omega, \eta_r \rangle + a(r) \).

Let \( \rho > r \), then \( a(\rho) + \langle \omega, x \rangle \leq u(x) \) in \( B_\rho(o) \). Taking \( x = r \eta_r \), we see that \( a(\rho) + r \langle \omega, \eta_r \rangle \leq u(r \eta_r) = a(r) + r \langle \omega, \eta_r \rangle \). This implies that \( a(r) \) is decreasing in \( r \). We now apply part (a). Let \( \eta \) be such that \( \eta_{r_k} \to \eta \) (choosing a subsequence of \( r_k \to \infty \) if necessary). Then

\[
\lim_{r_k \to \infty} \frac{u(r_k \eta_{r_k})}{r_k} = \langle \omega, \eta \rangle + \lim_{r_k \to \infty} \frac{a(r_k)}{r_k}.
\]

Using part (a) of the lemma, we see that \( \lim_{r_k \to \infty} a(r_k)/r_k = 0 \). Now suppose that \( \omega \) is the only limit point. Then it follows that \( \lim_{r \to \infty} u(r, \eta)/r = \langle \omega, \eta \rangle \).

The final conclusion now follows. \( \square \)

**Remark 2.** It is easy to show that if \( b(r) = \sup \{ c : c + \langle x, \omega \rangle \geq u(x), \ x \in \partial B_r(o) \} \), \( \omega \) as in Lemma 4, then \( \lim_{r \to \infty} b(r)/r_k = 0 \). We have \( a(r_k) + \langle \omega, x \rangle \leq u(x) \leq b(r_k) + \langle \omega, x \rangle, \ x \in B_{r_k}(o) \). If \( \omega \) is the only limit point then the above hold for \( r \to \infty \). It is unclear at this time whether any of these leads to \( u \) being affine.

**References**


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