Automatic Continuity of Separating Linear Isomorphisms on a Class of Topological Algebras

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Abstract

The continuity and general form of a separating linear isomorphisms $A : C(T) \to C(S)$, when $T$ and $S$ are compact topological spaces, have been studied in last few years. Recently, we have changed the conditions on $T$ and $S$ and proved that the separating linear isomorphism $A : C_c(T) \to C_c(S)$ has a general form. In this note, we prove the continuity of such isomorphisms.

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1 Introduction

When $T$ and $S$ are compact topological spaces, in [3] it is proved that every separating linear isomorphism $A : C(T) \to C(S)$ is continuous and has a general form. In [4], the continuity and general form of bi-separating linear maps between the algebras $B(E)$ of all linear continuous maps on a Banach space $E$ is also discussed, where by a bi-separating linear isomorphism we mean a separating linear isomorphism with a separating linear inverse. The continuity and general form of bi-separating linear isomorphisms between standard subalgebras of bounded operators on Frechet spaces is considered in [1]. In [2]The authors also have recently proved that, where $T$ is non-compact locally compact Hausdorff and $S$ is only Hausdorff space, then every separating linear isomorphism $A : C_c(T) \to C_c(S)$ has a general form. In this note we prove the continuity of such isomorphisms, if moreover $S$ is a locally compact and completely regular space.

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Let us recall that a linear map $A$ between the algebras $X$ and $Y$ is said to be separating if $xy = 0 \Rightarrow A(x)A(y) = 0$ for all $x, y \in X$.

## 2 The continuity of separating linear isomorphisms

Let $T$ be a non-compact, normal, locally compact and $S$ be a Hausdorff space. Suppose $A : C_c(T) \to C_c(S)$ is a linear separating map. In [2] it is proved that there are three subsets $S_1, S_2, S_3$ for $S$ such that $S = S_1 \cup S_2 \cup S_3$ where $S_3$ is closed and there is a continuous map $\varphi : S_1 \cup S_2 \to T$ and a non vanishing continuous map $\chi : S_1 \to \mathbb{C}$ such that for every $f \in C_c(T)$, and every $s \in S_1$:

$$Af(s) = \chi(s) \cdot f \circ \varphi(s),$$

and $Af \equiv 0$ on $S_3$. Moreover, for all $f \in C_c(T)$, if $(\text{supp} f) \cap \varphi(S_2) = \phi$, then $Af|_{S_2} \equiv 0$.

Here we prove that if moreover $S$ is a locally compact and completely regular space, then the separating linear isomorphism $A$ is continuous.

**Proposition 2.1** The two sets $S_1$ and $S_2$ have the following properties:

1. The subset $S_1$ of $S$ is closed.
2. $\varphi(S_2) \cap \text{int} K$ is finite for every compact subset $K$ of $T$.

**Proof.** Following the proof of [2; theorem 3.1], for $s \in S$, if we denote by $\delta_s$ the evaluation functional of $C_c(S)$ at the point $s$ then, for $f \in C_c(T)$, $\delta_s \circ A(f) = Af(s)$, and $S_3 = \{s \in S : \delta_s \circ A \equiv 0\}$, $S_2 = \{s \in S : \delta_s \circ A$ is discontinuous} and $S_1 = S \setminus (S_2 \cup S_3)$, and so $S_1 = \{s \in S : \delta_s \circ A$ is non-zero and continuous}.\]

To prove (1), let $(s_{\alpha})_{\alpha \in I}$ be a net in $S_1$ which converges to $s \in S$. Since $T$ is locally compact and $\varphi(s) \in T$, so $\exists f \in C_c(T); f(\varphi(s)) \neq 0$. By the continuity of the map $f$, we have $f(\varphi(s_{\alpha})) \to f(\varphi(s))$. Therefore, there exists $\alpha_0 \in I$ such that for all $\alpha$ with $\alpha \geq \alpha_0$ we have $f(\varphi(s_{\alpha})) \neq 0$. Now, $\chi(s_{\alpha}) \neq 0$ so $Af(s_{\alpha}) \neq 0$ for all $\alpha \geq \alpha_0$. By regarding to $C_c(T) = \ker \delta_{s_{\alpha}} \circ A \oplus \langle f \rangle$, for all $\alpha \geq \alpha_0$ we can assume the existence of an scalar $\beta_\alpha$ and a map $h_{\alpha} \in \ker \delta_{s_{\alpha}} \circ A$ for every $g \in C_c(T)$ such that $g = h_\alpha + \beta_\alpha f$, thus $g(\varphi(s_{\alpha})) = h_\alpha(\varphi(s_{\alpha})) + \beta_\alpha f(\varphi(s_{\alpha}))$. Since $s_{\alpha} \in S_1$, so $\ker \delta_{s_{\alpha}} \circ A = K_{s_{\alpha}}$ where $K_s = \{f \in C_c(T); f(\varphi(s)) = 0\}$ for every $s \in S$. We put $\beta_\alpha = \frac{g(\varphi(s_{\alpha}))}{f(\varphi(s_{\alpha}))}$, and therefore we have $g(\varphi(s_{\alpha})) = h_\alpha(\varphi(s_{\alpha})) + \beta_\alpha f(\varphi(s_{\alpha}))$. Now we put $\beta = \frac{Af(s)}{f(\varphi(s))}$, then we have $Ag(s) = \beta \cdot g(\varphi(s))$. In other words, we get $\delta_s \circ A(g) = \beta \cdot (g \circ \varphi)(s)$ for every $g \in C_c(T)$, and so $\delta_s \circ A$ is continuous, which means $s \in S_1$.

For (2), we suppose that there exists a compact subset $K$ of $T$ such that $\varphi(S_1) \cap \text{int} K$ is infinite. Since $K$ is compact so there exists a sequence $(\varphi(s_n))_{n \in \mathbb{N}}$.
of distinct elements of \( \text{int}K \) with \( s_n \in S_2 \) for \( n \in N \) and \( (\varphi(s_n))_{n \in N} \) converges to a point of \( K \). Therefore we can assume that \( (U_n)_{n \in N} \) is a pair wise disjoint sequence of open subsets of \( K \) such that \( \varphi(s_n) \in U_n \subseteq K \) for every \( n \in N \). We can also assume that the closure of every \( U_n \) is compact. Let \( V_n \) be a neighborhood of \( \varphi(s_n) \) with \( V_n \subseteq U_n \) and \( \overline{V_n} \) is compact. Thus there exists a map \( g_n \in C_c(T) \) such that \( 0 \leq g_n \leq 1 \), \( g_n|_{V_n} \equiv 1 \) and \( \text{coz}(g_n) \subseteq U_n \) for each \( n \in N \). On the other hand, since \( \delta_{s_n} \circ A \) is discontinuous, there exists a map \( h_n \in C_c(T) \) with \( \sup\{|h_n(t)| : t \in K\} \leq 1 \) and such that \( |\delta_{s_n} \circ A(h_n)| = |Ah_n(s_n)| \geq n^3 \) for all \( n \in N \). We put \( f_n = 1/n^2h_n \cdot g_n \). Since \( g_n \equiv 1 \) on \( V_n \), then we have that \( |Af_n(s_n)| = 1/n^2 \cdot |Ah_n(s_n)| > n \). So \( |Af_n(s_n)| > n \) for each \( n \in N \). By regarding to \( |f_n| \leq 1/n^2 \) we define \( f = \sum_{n \in N} f_n \). Since \( \text{supp} f \subseteq \cup \text{supp} f_n \subseteq K \), so \( f \in C_c(T) \). On the other hand, as \( (U_n) \) is pair wise disjoint and \( \text{coz}(f_n) \subseteq U_n \), for all \( n \in N \), then \( Af_n|_{\varphi^{-1}(U_m)} \equiv 0 \) for \( n \neq m \). Thus \( |Af(s_n)| = |Af_n(s_n)| > n \) for all \( n \in N \), which is a contradiction with \( Af \) is bounded.

**Theorem 2.2** Let \( A : C_c(T) \to C_c(S) \) be a separating linear isomorphism. Then \( A \) is continuous and \( Af(s) = \chi(s) \cdot f(\varphi(s)) \) for all \( s \in S \) and \( f \in C_c(T) \).

**Proof.** Since \( A \) is injective, by [2; corollary 3.3] \( \overline{\varphi(S)} = T \). Let \( t \in T \) be a limit point and \( U \) be a neighborhood of \( t \) with compact closure. By proposition (2.1), \( U \cap \varphi(S_2) \) is finite where \( U \cap \varphi(S) \) is infinite and therefore \( U \cap \varphi(S_1) \) is infinite. Since \( \varphi(S_2) \) consists only of the limit points of \( T \), this implies that \( \varphi(S_1) = T \).

For every \( f \in C_c(T) \) and \( s \in S_1 \) we have \( Af(s) = \chi(s) \cdot f(\varphi(s)) \). Now, if \( Af(s) = 0 \) for \( s \in S_1 \) and some \( f \in C_c(T) \), since \( \chi(s) \neq 0 \) so \( f(\varphi(s)) = 0 \). Therefore, if \( Af|_{S_1} = 0 \), then \( f(\varphi(S_1)) \equiv 0 \) i.e. \( f|_{\varphi(S_1)} \equiv 0 \) and since \( \varphi(S_1) = T \), so \( f = 0 \) on \( T \). Now, we claim \( S_2 = \emptyset \). For if \( s \in S_2 \), then \( \{s\} \cap S_2 = \emptyset \) and since \( S_1 \) is closed by Urysohn’s lemma there exists \( g \in C_c(S) \) with \( g(s_0) = 1 \) and \( g = 0 \) on \( S_1 \). Since \( A \) is bijection and \( g \) is a non-zero, so there exists a non-zero \( f \) in \( C_c(T) \) with \( Af = g \), i.e. \( Af(s) = 1 \) and \( Af|_{S_1} \equiv 0 \) which is a contradiction.

We close the paper with a remark which says that if we consider \( A : C_0(T) \to C_0(S) \) with the same conditions then \( A \) has a general form, but we believe the continuity of \( A \) is still open.

**Remark.** Let \( A : C_0(T) \to C_0(S) \) be a linear separating map. Suppose \( A_1 = A|_{C_0(T)} \) and \( S_0 = \{s \in S : \delta \circ A \text{ is continuous on } C_0(T)\} \). Since the proof of Theorem (3.1) of [2] valid if we replace \( C_c(S) \) by \( C_0(S) \), so there exists a continuous map \( \varphi : S_1 \cup S_2 \to T \) and a non vanishing continuous map \( \chi : S_1 \to C \) such that \( A_1 f(s) = \chi(s) \cdot f(\varphi(s)) \) for every \( f \in C_c(T), s \in S_0 \). Now, let \( f \in C_0(T) \), and choose \( (f_n) \) from \( C_c(T) \) with \( f_n \to f \) in \( C_0(T) \) and let
$s \in S_0$. Then $\delta_s \circ A(f_n) \to \delta_s \circ A(f)$ and $\delta_s \circ A(f) = \lim_{n \to \infty} \chi(s) \cdot f_n(\varphi(s)) = \chi(s) \cdot f(\varphi(s))$.

References


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