Generalization of Reich’s Fixed Point Theorem

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Abstract

In this paper, we present generalizations of some fixed point theorems using the notion of w-distance on a metric space. The results herein contain the work of many authors including Reich, Morales, Rakotch, Chu and Diaz.

Mathematics Subject Classification: 54H25, 54A40

Keywords: fixed points, contraction, w-distance

1 Introduction

The notion of w-distance on a metric space was introduced by Kada, Suzuki and Takahashi in [3]. In [9], various properties and examples of w-distances together with some fixed point theorems are given in terms of w-distances. In [7], T. Suzuki gave another property of w-distance which generalize some of the results in [3] and proved several fixed point theorems which are generalization of Banach contraction principle[1] and Kannan[4] fixed point theorems. Moreover, characterization of metric completeness is also discussed.

In this paper, we prove some fixed point theorems which are generalizations of Reich’s theorem[7] and the results of Morales[5].
2 Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the set of real numbers and $\mathbb{R}^+ = [0, +\infty)$.

**Definition 2.1** [5] Let $(M, d)$ be a metric space. Then the function $p : M \times M \rightarrow [0, +\infty)$ is called a $w$-distance on $M$ if the following conditions are satisfied

$w_1$. $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in M$

$w_2$. for any $x \in M$, $p(x, .) : M \rightarrow [0, +\infty)$ is lower semi-continuous

$w_3$. for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

**Example 2.2** For $(R, d)$, a metric space, where $d$ is usual metric on $R$ and $a > 0$, define $p_1(x, y) = |x - y|$ and $p_2(x, y) = a$ for all $x, y \in R$. Clearly $p_1$ and $p_2$ satisfies $w_1$-$w_3$. Some other examples of $w$-distances are given in [7].

The following results are crucial in the proofs of our theorems.

**Lemma 2.3** (5) Let $(M, d)$ be a metric space and $p$ be a $w$-distance on $M$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, +\infty)$ converging to zero and $x, y, z \in M$. Then the following hold.

a. If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$ then $y = z$.

b. If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to $z$.

c. If $p(x_n, x_m) \leq \alpha_n$ for any $m, n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a cauchy sequence.

d. If $p(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a cauchy sequence.

**Definition 2.4** [5] Let $(M, d)$ be a metric space and $p$ be a $w$-distance on $M$. Denote by $\mathcal{F}$ the family of functions $\alpha(x, y)$ satisfying the following conditions

i. $\alpha(x, y) = \alpha(p(x, y))$; i.e. $\alpha$ is dependent on the $w$-distance $p$ on $M$.

ii. $0 \leq \alpha(p) < 1$ for every $p > 0$.

iii. $\alpha$ is monotonically decreasing function of $p$.

**Definition 2.5** Let $(M, d)$ be a metric space and $p$ be a $w$-distance on $M$. A mapping $T : M \rightarrow M$ is called a $w$-Reich contraction if there exists a function $p$ such that $p(Tx, Ty) \leq a(x, y)p(x, Tx) + b(x, y)p(y, Ty) + c(x, y)p(x, y)$ for all $x, y \in M$, where $a(x, y), b(x, y), c(x, y) \in \mathcal{F}$ and $a(x, y) + b(x, y) + c(x, y) < 1.$
Remark 2.6 If $p = d$ then $T$ is called Reich’s contraction[7].

Remark 2.7 If $p = d$ and $a(x,y)=0$, $b(x,y)=0$ then $T$ is called Rakotch’s contraction[6].

3 Main Results

In this section we next generalize the Reich’s theorem [7] and the results in [5].

Theorem 3.1 Let $(M,d)$ be a complete metric space and let $p$ be a $w$-distance on $M$ and $T : M \to M$ be a $w$-Reich contraction. Then there exists a unique $z \in M$ such that $Tz = z$. Further $z$ satisfies $p(z, z) = 0$.

Proof: Since $T$ is a $w$-Reich contraction there exist functions $a(x,y), b(x,y), c(x,y) \in \mathbb{F}$ such that

$$p(Tx,Ty) \leq a(x,y)p(x,Tx) + b(x,y)p(y,Ty) + c(x,y)p(x,y)$$

for all $x,y \in M$.

Let $x_0 \in M$ and define $x_n = T^n x_0$, $n \in \mathbb{N}$. Then

$$p(x_n, x_{n+1}) = p(Tx_n, Tx_{n+1})$$

and

$$p(x_n, x_{n+1}) = a(x_{n-1}, x_n)p(x_{n-1}, Tx_{n-1}) + b(x_{n-1}, x_n)p(x_n, Tx_n) + c(x_{n-1}, x_n)p(x_{n-1}, x_n)$$

for all $n \geq 0$.

Next, if $p(x_k, x_{k+1}) \geq \varepsilon_0$; $k=0, 1, \ldots, n-1$ for $\varepsilon_0 > 0$, then by monotonicity of $\alpha$, $\alpha(p(x_k, x_{k+1})) \leq \alpha(\varepsilon_0)$ and hence $p(x_n, x_{n+1}) \leq \alpha^n(\varepsilon_0)p(x_0, Tx_0)$, but $0 \leq \alpha^n(\varepsilon_0) < 1$.

Therefore, by Lemma 2.3, we have $\lim_{n \to \infty}p(x_n, x_{n+1}) = 0$.

We shall show that $\{x_n\}$ is a Cauchy sequence in $(M,d)$.

For $m > 0$, $p(x_n, x_m) \leq \prod_{k=0}^{n-1} \alpha(p(x_k, x_{k+m}))p(x_0, Tx_0)$. If $p(x_k, x_{k+m}) \geq \varepsilon_0$ for given $\varepsilon_0 > 0$ and $k=0, 1, \ldots, n-1$, then $p(x_n, x_{n+m}) \leq \alpha^n(\varepsilon_0)p(x_0, Tx_0) \to 0$ as $n \to \infty$ and by Lemma 2.3, we have that $\{x_n\}$ is a Cauchy sequence. Since $(M,d)$ is complete, $\{x_n\}$ converges to some $z \in M$. Since $x_m \to z$ and $p(x_n, \cdot)$ is lower semi continuous, therefore $p(x_n, z) \leq \lim_{n \to \infty}p(x_n, x_m) \leq \alpha^n(\varepsilon_0)p(x_0, Tx_0)$. So $\lim_{n \to \infty}p(x_n, z) = 0$.

On the other hand

$$p(x_n, Tz) = p(Tx_n, Tz) \leq a(x_{n-1}, z)p(x_{n-1}, x_n) + b(x_{n-1}, z)p(z, Tz) + c(x_{n-1}, z)p(x_{n-1}, z)$$

and

$$p(Tx_n, Tz) \leq a(x_{n-1}, z)p(x_{n-1}, Tz) + b(x_{n-1}, z)p(z, Tz) + c(x_{n-1}, z)p(x_{n-1}, z)$$

As $n \to \infty$ and $\alpha < 1$ so $\lim_{n \to \infty}p(x_n, Tz) = 0$ and by Lemma 2.3 we have $Tz = z$.

Now $p(z,z) = p(Tz,Tz) \leq a(z,z)p(z, Tz) + b(z,z)p(z, Tz) + c(z,z)p(z, z) < p(z, z)$. 
So \( p(z,z)=0 \). If \( y=Ty \) then
\[
p(z,y)=p(Tz,Ty) \leq a(z,y)p(z,Tz)+b(z,y)p(y,Ty)+c(z,y)p(z,y)<p(z,y)
\]
and \( p(z,y)=0 \).
Also \( p(y,z)=p(Ty,Tz) \leq a(y,z)p(y,Ty)+b(y,z)p(z,Tz)+c(y,z)p(y,z)<p(y,z) \)
so \( p(y,z)=0 \). Hence by (a) of Lemma 2.3 we have \( z=y \).

**Remark 3.2** In case, \( a(x,y)=0, b(x,y)=0 \) in Theorem 3.1 then we have Theorem 1 of [3]. Further if we take \( p=d \), \((M,d)\) a complete metric space then we get the Rakotch’s theorem [4] and if \( c(x,y)=k \), \( 0 \leq k<1 \), we get Banach fixed point theorem[1].

**Theorem 3.3** Let \((M,d)\) be a complete metric space and let \( p \) be a \( w \)-distance on \( M \). A mapping \( T : M \rightarrow M \) is such that for some \( m \in \mathbb{N} \), \( T^m \) is a \( w \)-Reich contraction. Then \( T \) has a unique fixed point, i.e. there exists a unique \( z \in M \) such that \( Tz=z \). Further \( z \) satisfies \( p(z,z)=0 \).

**Proof:** Since for some \( n \in \mathbb{N} \), \( T^m \) is a \( w \)-Reich contraction there exists mappings \( a(x,y), b(x,y), c(x,y) \in \mathfrak{F} \) such that
\[
p(T^nx,T^ny) \leq a(x,y)p(x,T^nx)+b(x,y)p(y,T^ny)+c(x,y)p(x,y)
\]
for all \( x,y \in M \).

Hence by Theorem 3.1, there exists a unique \( z \in M \) such that \( T^mz=z \) for \( m \in \mathbb{N} \) and \( Tz=T(T^mz)=T^m(Tz) \) it follows that \( z=Tz \).

**Corollary 3.4** Let \((M,d)\) be a complete metric space and let \( p \) be a \( w \)-distance on \( M \) and \( T : M \rightarrow M \) is a mapping such that for some \( m \in \mathbb{N} \), \( T^m \) is a \( w \)-Reich contraction. Then \( T \) has a unique fixed point, i.e. there exists a unique \( z \in M \) such that \( Tz=z \). Further \( z \) satisfies \( p(z,z)=0 \).

**Remark 3.5** In Theorem 3.3, if we take \( p=d \) and \( a(x,y)=0, b(x,y)=0, c(x,y)=k \), \( 0 \leq k<1 \), then we get the Chu-Diaz’s Theorem [2].

**Theorem 3.6** Let \( M \) be a non empty set, \( d \) and \( \rho \) be two metrics on \( M \), \( p \) and \( \rho \) their respective \( w \)-distances on \( M \) and \( T : M \rightarrow M \) be a mapping. Suppose that:

a. \( p(x,y) \leq \rho(x,y) \) for all \( x,y \in M \).

b. \((M,d)\) is a complete metric space.

c. \( T:(M,\rho)\rightarrow(M,\rho) \) is a \( w \)-Reich’s contraction.

Then there exists \( z \in M \) such that \( Tz=z \) and moreover \( p(z,z)=0 \).

**Proof:** Let \( x_o \in M \) and define \( x_n=T^n x_o, \ n \in \mathbb{N} \). From (c), \( \{x_n\} \) is a cauchy sequence in \((M,\rho)\). By (a) and Lemma 2.3, \( \{x_n\} \) is a cauchy sequence in \((M,d)\) and by (b) it converges. The rest of the proof is similar to Theorem 3.1.
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References

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Received: September, 2008