On the Convergence of Three-Step Iterations in the Class of Zamfirescu Operators

Shaini Pulickakunnel 1 and Neeta Singh

Department of Mathematics
University of Allahabad
Allahabad-211002, India

Abstract

In this paper a strong convergence theorem is established for a three-step iterative scheme for the class of Zamfirescu operators in normed linear spaces. The result obtained in this paper extends and improves upon, among others, the corresponding result of V. Berinde (2004) [1].

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1 Introduction and preliminary definitions

One of the most studied class of quasi-contractive type operators is that of Zamfirescu operators. Recently several authors have studied multi-step iteration schemes to approximate fixed points of various classes of mappings in Banach spaces. In this paper a new three-step iteration scheme is introduced and a strong convergence theorem is proved for the class of Zamfirescu operators. The scheme is defined as follows.

Let $X$ be a normed space, $C$ a nonempty closed convex subset of $X$, and $T : C \to C$ be a given mapping. Then for a given $x_1 \in C$, compute the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$
\begin{align*}
    z_n &= a_n Tx_n + (1 - a_n)x_n \\
    y_n &= b_n Tz_n + c_n Tx_n + (1 - b_n - c_n)x_n \\
    x_{n+1} &= \alpha_n Ty_n + \beta_n Tz_n + \gamma_n Tx_n + (1 - \alpha_n - \beta_n - \gamma_n)x_n, \quad n \geq 1
\end{align*}
$$

(1)

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are appropriate sequences in $[0, 1]$.

1Corresponding author.
E-mail address: shainipv@gmail.com (Shaini P.), n.s32132@yahoo.com (N. Singh).
If \( a_n = c_n = \beta_n = \gamma_n = 0 \), then (1) reduces to the Ishikawa iterative scheme
\[
\begin{align*}
  y_n &= b_nTx_n + (1 - b_n)x_n \\
  x_{n+1} &= \alpha_nTy_n + (1 - \alpha_n)x_n, \quad n \geq 1
\end{align*}
\]
where \( b_n \) and \( \alpha_n \) are appropriate sequences in \([0, 1]\).

If \( a_n = b_n = c_n = \beta_n = \gamma_n = 0 \), then (1) reduces to the Mann iterative scheme
\[
x_{n+1} = \alpha_nTx_n + (1 - \alpha_n)x_n, \quad n \geq 1
\]
where \( \alpha_n \) is an appropriate sequence in \([0, 1]\).

We recall the following definitions in a metric space \((X, d)\).

A mapping \( T : X \to X \) is called an \( a \)-contraction if

\[ (z_1) \quad d(Tx, Ty) \leq ad(x, y) \text{ for all } x, y \in X, \text{ where } a \in [0, 1). \]

The map \( T \) is called Kannan mapping \([3]\) if there exists \( b \in [0, \frac{1}{2}) \) such that

\[ (z_2) \quad d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X. \]

A similar definition is due to Chatterjea \([2]\): there exists \( c \in [0, \frac{1}{2}) \) such that

\[ (z_3) \quad d(Tx, Ty) \leq c[d(x, Tx) + d(y, Tx)] \text{ for all } x, y \in X. \]

It is known, see Rhoades \([7]\) that \((z_1), (z_2)\), and \((z_3)\) are independent contractive conditions. An operator \( T \) which satisfies at least one of the contractive conditions \((z_1) - (z_3)\) is called a Zamfirescu operator or a \( Z \)-operator. Alternatively we say that \( T \) satisfies Condition \( Z \).

The purpose of this paper is to establish a strong convergence theorem to approximate fixed points of \( Z \)-operators in normed spaces through the three-step iteration defined by (1). Our result extends and improves the corresponding result of Berinde \([1]\), and others.

We need the following lemma, which is proved in \([4]\).

**Lemma 1.1.** Let \( \{r_n\}, \{s_n\}, \{t_n\} \) and \( \{k_n\} \) be sequences of nonnegative numbers satisfying

\[
r_{n+1} \leq (1 - s_n)r_n + s_nt_n + k_n, \quad \text{for all } n \geq 1.
\]

If \( \sum_{n=1}^{\infty} s_n = \infty \), \( \lim_{n \to \infty} t_n = 0 \) and \( \sum_{n=1}^{\infty} k_n < \infty \) hold, then \( \lim_{n \to \infty} r_n = 0. \)

2 Main result

**Theorem 2.1.** Let \( C \) be a nonempty closed convex subset of a normed linear space \( X \). Let \( T : C \to C \) be a \( Z \)-operator with \( F(T) \neq \emptyset \), where \( F(T) \) is the set of fixed points of \( T \). For a given \( x_1 \in C \), let \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) be the sequences defined as in (1), where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \) and \( \{c_n\} \) be real sequences in \([0, 1]\) such that \( b_n + c_n \) and \( \alpha_n + \beta_n + \gamma_n \) are in \([0, 1]\) for all
Convergence of three-step iterations

\[ n \geq 1 \text{ and satisfying at least one of the following conditions:} \]
\[ (i) \sum \alpha_n = \infty, \quad (ii) \sum \beta_n = \infty, \quad (iii) \sum \gamma_n = \infty, \quad (iv) \sum \alpha_n b_n = \infty, \]
\[ (v) \sum \alpha_n c_n = \infty, \quad (vi) \sum \alpha_n \beta_n = \infty, \quad (vii) \sum \alpha_n a_n b_n = \infty. \]

Then \( \{x_n\} \) converges strongly to a fixed point of \( T \).

**Proof.** It follows from \( F(T) \neq \emptyset \), that \( T \) has a fixed point in \( C \), say \( p \). Consider \( x, y \in C \). Since \( T \) is a Zamfirescu operator, at least one of the conditions (\( z_1 \)), (\( z_2 \)) and (\( z_3 \)) is satisfied. If (\( z_2 \)) holds, then
\[
\|Tx - Ty\| \leq b[\|x - Tx\| + \|y - Ty\|] \\
\leq b\{\|x - Tx\| + \|y - x\| + \|x - T_x\| + \|T x - Ty\|}\]
which implies
\[
(1 - b) \|Tx - Ty\| \leq b \|x - y\| + 2b \|x - Tx\|
\]
since \( 0 \leq b < 1 \) we get
\[
\|Tx - Ty\| \leq \frac{b}{1 - b} \|x - y\| + \frac{2b}{1 - b} \|x - Tx\|. \tag{3}
\]
If (\( z_3 \)) holds, then similarly we obtain
\[
\|Tx - Ty\| \leq \frac{c}{1 - c} \|x - y\| + \frac{2c}{1 - c} \|x - Tx\|. \tag{4}
\]
Denote
\[
\delta = \max \left\{ a, \frac{b}{1 - b}, \frac{c}{1 - c} \right\}. \tag{5}
\]
Then we have \( 0 \leq \delta < 1 \) and, in view of (\( z_1 \)), (\( z_2 \)), (\( z_3 \)) and (\( z_4 \)) it results that the inequality
\[
\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\| \tag{6}
\]
holds for all \( x, y \in C \).

With \( y = x_n, y_n, z_n \) and \( x = p \) in (\( 6 \)) we obtain the following inequalities
\[
\|Tx_n - p\| \leq \delta \|x_n - p\| \tag{7}
\]
\[
\|Ty_n - p\| \leq \delta \|y_n - p\| \tag{8}
\]
\[
\|Tz_n - p\| \leq \delta \|z_n - p\| \tag{9}
\]
Now let \( \{x_n\} \) be the iteration process defined by (1) and \( x_1 \in C \) be arbitrary. Then
\[
\|x_{n+1} - p\| = \|\alpha_n Ty_n + \beta_n Tz_n + \gamma_n Tx_n + (1 - \alpha_n - \beta_n - \gamma_n)x_n - p\|
\]
\[
= \|\alpha_n(Ty_n - p) + \beta_n(Tz_n - p) + \gamma_n(Tx_n - p) + (1 - \alpha_n - \beta_n - \gamma_n)(x_n - p)\|
\]
\[
\leq \alpha_n \delta \|y_n - p\| + \beta_n \delta \|z_n - p\| + (1 - \alpha_n - \beta_n - \gamma_n)\|x_n - p\|
\]
by (7, 8&9) (10)

Again by (1) we have
\[
\|y_n - p\| = \|b_n Tz_n + c_n Tx_n + (1 - b_n - c_n)x_n - p\|
\]
\[
= \|b_n(Tz_n - p) + c_n(Tx_n - p) + (1 - b_n - c_n)(x_n - p)\|
\]
\[
\leq b_n \delta \|z_n - p\| + (1 - b_n - c_n + c_n \delta) \|x_n - p\|
\]
by (7, &9) (11)

Further it follows from (1) that
\[
\|z_n - p\| = \|a_n(Tx_n - p) + (1 - a_n)(x_n - p)\|
\]
\[
\leq (1 - a_n + a_n \delta) \|x_n - p\|
\]
by (7) (12)

Now from (11) and (12) we get
\[
\|y_n - p\| \leq b_n \delta (1 - a_n + a_n \delta) \|x_n - p\| + (1 - b_n - c_n + c_n \delta) \|x_n - p\|
\]
\[
\leq (1 - b_n - c_n + c_n \delta + b_n \delta - a_n b_n \delta + a_n b_n \delta^2) \|x_n - p\|
\]
(13)

Substituting (12) and (13) into (10) we obtain that
\[
\|x_{n+1} - p\| \leq \left\{\alpha_n \delta (1 - b_n - c_n + c_n \delta + b_n \delta - a_n b_n \delta + a_n b_n \delta^2) + \beta_n \delta (1 - a_n + a_n \delta) + (1 - \alpha_n - \beta_n - \gamma_n + \gamma_n \delta)\right\} \|x_n - p\|
\]

Simplifying we have
\[
\|x_{n+1} - p\| \leq \left\{1 - (\alpha_n a_n b_n \delta^2 + \alpha_n b_n \delta + \alpha_n c_n \delta + a_n \beta_n \delta + \alpha_n + \beta_n + \gamma_n)(1 - \delta)\right\} \|x_n - p\|.
\]

It follows from the given conditions and the Lemma 1.1 that \( \lim_{n \to \infty} \|x_n - p\| = 0. \)
Which implies that \( x_n \to p \in F(T) \). Hence the proof is complete. \( \square \)

**Corollary 2.2.** Let \( C \) be a nonempty closed convex subset of a normed space \( X \). Let \( T : C \to C \) be a Z-operator. For any initial guess \( x_1 \) in \( C \), let \( \{x_n\} \) be defined by (2) with the restriction that \( \sum_{n=0}^{\infty} \alpha_n = \infty \). If \( F(T) \neq \phi \), where \( F(T) \) is the set of fixed points of \( T \), then \( \{x_n\} \) converges strongly to a fixed point of \( T \).
Remark 2.3. (1). The Ishikawa iteration process being a special case of the new iteration scheme (1), Berinde’s theorem [1, Theorem 2] follows as a corollary from Theorem 2.1.
(2). Theorem 2.1 generalizes, Theorem 4 of Rhoades [5], and Theorem 8 of Rhoades [6] in the context of Mann iteration and Ishikawa iteration respectively on a uniformly convex Banach space, to the setting of normed spaces and simultaneously extends to the new iteration process (1).
(3). Chatterjea’s and Kannan’s contractive conditions \((z_2)\) and \((z_3)\) are both included in the class of Zamfirescu operators and so their convergence theorems for the Ishikawa iteration process are obtained in Corollary 2.2.

References


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