Abstract

In this paper, we introduce the notions of fuzzy $\gamma$-I-open sets and fuzzy $\gamma$-I-continuous functions to obtain a decomposition of fuzzy semi-I-continuity. We also investigated the fundamental properties of such functions.

Keywords: fuzzy ideal, fuzzy $\gamma$-I-open set, fuzzy $\gamma$-I-continuous, fuzzy $\delta$-I-open set.

1 Introduction and Preliminaries

Fuzziness is one of the most important and useful concepts in the modern scientific studies. This is because of the fact that since Zadeh [14] first introduced the notion of fuzzy sets applications of this idea was made by many authors. Kuratowski [8], Jankovic [7] and several other authors studied on the importance of ideal in general topology. The concept of topology of fuzzy sets may be relevant to quantum particles physics particularly in connection with string theory and E-infinity theory [4, 5]. Mahmoud [9] and Sarkar [12] independently presented some of the ideal concepts in fuzzy trend and studied many of their properties. Recently, Hatir [6], Gupta and Rajneesh [3] and Nasef [10] introduced fuzzy semi-I-open sets, weakly fuzzy semi-I-open sets and fuzzy pre-I-open sets, respectively and also obtained a decomposition of fuzzy continuity.

Throughout this paper, $X$ represents a nonempty fuzzy set and fuzzy subset $A$ of $X$, denoted by $A \leq X$, is characterized by a membership function in the sense of Zadeh [14]. The basic fuzzy sets are the empty set, the whole
set and the class of all fuzzy subsets of $X$ which will be denoted by $0, 1$ and $I^X$, respectively. A subfamily $\tau$ of $I^X$ will denote topology of fuzzy sets on $I^X$ as defined by Chang [2]. The collection of all fuzzy open sets containing $x$ will be denoted by $\tau(x)$. By $(X, \tau)$, we mean a fuzzy topological space in Chang's sense. A fuzzy point in $X$ with support $x \in X$ and value $\alpha (0 < \alpha \leq 1)$ is denoted by $x_\alpha$. For a fuzzy subset $A$ of $X$, $\text{Cl}(A)$, $\text{Int}(A)$ and $1 - A$ will, respectively, denote the closure, interior and complement of $A$. A nonempty collection $I$ of fuzzy subsets of $X$ is called a fuzzy ideal [12] if and only if

1. $B \in I$ and $A \leq B$, then $A \in I$ (heredity),
2. if $A \in I$ and $B \in I$ then $A \lor B \in I$ (finite additivity).

An fuzzy ideal topological space, denoted by $(X, \tau, I)$ means a fuzzy topological space with a fuzzy ideal $I$ and fuzzy topology $\tau$. For $(X, \tau, I)$, the fuzzy local function of $A \leq X$ with respect to $\tau$ and $I$ is denoted by $A^*(\tau, I)$ (briefly $A^*$) and is defined as $A^*(\tau, I) = \vee \{x \in X : A \leq U \land U \notin I \text{ for every } U \in \tau(x)\}$. While $A^*$ is the union of the fuzzy points $x$ such that if $U \in \tau(x)$ and $E \in I$, then there is at least one $y \in X$ for which $U(y) + A(y) - 1 > E(y)$. Fuzzy closure operator of a fuzzy set in $(X, \tau, I)$ is defined as $\text{Cl}^*(A) = A \lor A^*$. In $(X, \tau, I)$, the collection $\tau^*(I)$ means an extension of fuzzy topological space than $\tau$ via fuzzy ideal which is constructed by considering the class $\beta = \{U - E : U \in \tau, E \in I\}$ as a base[12]. This topology of fuzzy sets is considered as generalization of the ordinary one.

First, we shall recall some definitions used in the sequel.

**Lemma 1.1** [12] Let $(X, \tau, I)$ be fuzzy ideal topological space and $A, B$ be fuzzy subsets of $X$. The following properties hold:

1. if $A \leq B$, then $A^* \leq B^*$,
2. $\text{Cl}(A^*) \leq \text{Cl}(A)$,
3. if $U \in \tau$, then $U \land A^* \leq (U \land A)^*$.

**Definition 1.1** A fuzzy subset $A$ of a fuzzy ideal topological space $(X, \tau, I)$ is said to be

1. fuzzy semi-I-open [6] if $A \leq \text{Cl}^*(\text{Int}(A))$,
2. fuzzy $\alpha$-I-open [13] if $A \leq \text{Int}(\text{Cl}^*(\text{Int}(A)))$,
3. weakly fuzzy semi-I-open [3] if $A \leq \text{Cl}^*(\text{Int}(\text{Cl}(A)))$,
4. fuzzy pre-I-open [10] if $A \leq \text{Int}(\text{Cl}^*(A))$,

**Definition 1.2** [1] Any subclass $\mathcal{U}$ of $I^X$ is called supratopology on $X$ if $\mathcal{U}$ contains $0, 1$ and closed under arbitrary union.
2 Fuzzy $\gamma$-I-open sets

Definition 2.1 A fuzzy subset $A$ of a fuzzy ideal topological space $(X, \tau, I)$ is said to be fuzzy $\gamma$-I-open set if

$$A \leq \text{Cl}^*(\text{Int}(A)) \lor \text{Int}(\text{Cl}^*(A)).$$

The complement of fuzzy $\gamma$-I-open set will be called fuzzy $\gamma$-I-closed set. The collection of all fuzzy $\gamma$-open (resp. fuzzy $\gamma$-I-closed) sets in $(X, \tau, I)$ will be denoted by $\text{FGIO}(X)$ (resp. $\text{FGIC}(X)$).

Theorem 2.1 In fuzzy ideal topological space $(X, \tau, I)$, the following statements hold:

1. every fuzzy open set is fuzzy $\gamma$-I-open set,
2. every fuzzy semi-I-open set is fuzzy $\gamma$-I-open set,
3. every fuzzy pre-I-open set is fuzzy $\gamma$-I-open set.

Proof. It is easy and therefore omitted.

Remark 2.1 For several sets defined above, we have the following implications:

$$\begin{align*}
\text{fuzzy open} & \implies \text{fuzzy } \alpha\text{-I-open} \implies \text{fuzzy semi-I-open} \\
\Downarrow & \quad \Downarrow \\
\text{fuzzy I-open} & \implies \text{fuzzy pre-I-open} \implies \text{fuzzy } \gamma\text{-I-open}
\end{align*}$$

Example 2.1 Let $X = \{a, b, c\}$ and $A, B, C$ be fuzzy subsets of $X$ defined as fellows:

- $A(a) = 0.5, \quad A(b) = 0.4, \quad A(c) = 0.6,$
- $B(a) = 0.5, \quad B(b) = 0.5, \quad B(c) = 0.6.$

Let $\tau = \{0, A, 1\}$. If we take $I = \{0\}$, then $B \in \text{FGIO}(X)$, but $B$ is not fuzzy open, since $B^* = \text{Cl}(B)$.

Example 2.2 Let $X = [0, 1]$ and $A, B, C$ be fuzzy subsets of $X$ defined as follows:

$A(x) = \begin{cases} 
x & \text{if } 0 \leq x \leq \frac{1}{2} \\
1 - x & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}$

$B(x) = \begin{cases} 
1 - 2x & \text{if } 0 \leq x < \frac{1}{2} \\
\frac{1}{2} & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}$
Proof. Since
for each $\alpha \in \Delta$. Theorem 2.2 Let $X = \{a, b, c\}$ and $A, B, C$ be fuzzy subsets of $X$ defined as follows:

$$
A(a) = 0.2, \quad A(b) = 0.3, \quad A(c) = 0.7,
B(a) = 0.1, \quad B(b) = 0.2, \quad B(c) = 0.2.
$$

Let $\tau = \{0, B, C, 1\}$. If we take $I = \{0\}$, then $A \in FGIO(X)$, but $A$ is not fuzzy pre-$I$-open.

Example 2.3 Let $X = \{a, b, c\}$ and $A, B, C$ be fuzzy subsets of $X$ defined as follows:

$$
A(a) = 0.2, \quad A(b) = 0.3, \quad A(c) = 0.7,
B(a) = 0.1, \quad B(b) = 0.2, \quad B(c) = 0.2.
$$

Let $\tau = \{0, B, C, 1\}$. If we take $I = \{0\}$, then $A \in FGIO(X)$, but $A$ is not fuzzy pre-$I$-open.

Theorem 2.2 Let $(X, \tau, I)$ be fuzzy ideal topological space. Let $A, U$ and $A_\alpha (\alpha \in \Delta)$ be fuzzy subsets of $X$. Then

1. if $A_\alpha$ is fuzzy $\gamma$-$I$-open set for each $\alpha \in \Delta$, then $\forall \alpha \in \Delta A_\alpha$ is fuzzy $\gamma$-$I$-open set,

2. if $A$ is fuzzy $\gamma$-$I$-open set and $U$ is fuzzy open set i.e. $U \in \tau$, then $U \land A$ is fuzzy $\gamma$-$I$-open set.

Proof. Since $A_\alpha$ is fuzzy $\gamma$-$I$-open set for each $\alpha \in \Delta$, we have

$$
A \leq Cl^*(Int(A)) \lor Int(Cl^*(A))
$$

for each $\alpha \in \Delta$. Thus by using Lemma 1.1, $\forall \alpha \in \Delta A_\alpha \leq \forall \alpha \in \Delta (Cl^*(Int(A_\alpha)) \lor Int(Cl^*(A_\alpha))) = \forall \alpha \in \Delta Cl^*(Int(A_\alpha)) \lor \forall \alpha \in \Delta Int(Cl^*(A_\alpha)) \leq Cl^*(\forall \alpha \in \Delta A_\alpha) \lor Int(Cl^*(\forall \alpha \in \Delta A_\alpha)).$ Hence $\forall \alpha \in \Delta A_\alpha$ is fuzzy $\gamma$-$I$-open set.

(2) Since $U \in \tau$ and $A \in FGIO(X)$ in $(X, \tau, I)$, then $U \land A \leq U \land (Cl^*(Int(A)) \lor Int(Cl^*(A))) = U \land Cl^*(Int(A)) \lor U \land Int(Cl^*(A)) \leq (Cl^*(Int(U \land A)) \lor Int(Cl^*(U \land A)).$ Hence $U \land A$ is fuzzy $\gamma$-$I$-open set.

Theorem 2.3 For a fuzzy ideal topological space $(X, \tau, I)$, the class of fuzzy $\gamma$-$I$-open sets forms a fuzzy supratopology.

Proof. This follows by using Theorem 2.2, therefore proof is omitted.

Definition 2.2 [11] For a space $(X, \tau, I)$, if $A \leq X$, the fuzzy restriction of $I$ to $A$, which is denoted by $I|_A$ and defined as: $I|_A = \{E \land A : E \in I\}$, where $I|_A$ is fuzzy ideal. By the fuzzy restriction of $I$ to $A$, the fuzzy relative topology of $\tau$ on $A$, denoted by $\tau|_A = \{A \land U : U \in \tau\}$. 

Lemma 2.1 [11] Let \((X, \tau, I)\) be space with \(A \leq B \leq X\). Then \(A^*(I_B, \tau_B) = A^*(I, \tau) \cap B\).

Theorem 2.4 Let \((X, \tau, I)\) be a fuzzy ideal topological space. If \(W \leq U \in \tau\) and \(W \in FGIO(X)\); then \(U \cap W \in FGIO(U, I|U, \tau|U)\).

Proof. Let \(W\) be a fuzzy \(\gamma\)-I-set in \((X, \tau, I)\). Since \(U\) is fuzzy open, we have \(\text{Int}U(W) = \text{Int}(W)\). By using this fact and Lemma 2.1, we have \(U \cap W \in FGIO(U, I|U, \tau|U)\).

Theorem 2.5 If a fuzzy subset \(A\) of a fuzzy ideal topological space \((X, \tau, I)\) is fuzzy \(\gamma\)-I-closed, then

\[\text{Cl}^*(\text{Int}(A)) \cap \text{Int}^*(\text{Cl}(A)) \leq A.\]

Proof. Since \(A \in FGIC(X)\), \(1 - A \in FGIO(X)\). We have

\[1 - A \leq \text{Cl}^*(\text{Int}(1 - A)) \cap \text{Int}^*(\text{Cl}(1 - A)) \leq \text{Cl}(\text{Int}(1 - A)) \cap \text{Int}(\text{Cl}(1 - A)) = (1 - \text{Cl}(\text{Int}(A))) \cap (1 - \text{Int}(\text{Cl}(A))) \leq (1 - \text{Cl}^*(\text{Int}(A))) \cap (1 - \text{Int}^*(\text{Cl}(A))) = 1 - \text{Cl}^*(\text{Int}(A)) \cap \text{Int}^*(\text{Cl}(A)).\]

Therefore, we obtain

\[\text{Cl}^*(\text{Int}(A)) \cap \text{Int}^*(\text{Cl}(A)) \leq A.\]

3 Decomposition of fuzzy continuity

Definition 3.1 A fuzzy subset \(A\) of \((X, \tau, I)\) is called fuzzy \(\delta\)-I-open if

\[\text{Int}(\text{Cl}^*(A)) \leq \text{Cl}^*(\text{Int}(A)).\]

Definition 3.2 A fuzzy subset \(A\) of \((X, \tau, I)\) is called fuzzy \(*\)-I-dense in itself if

\[A \leq A^*.\]

Definition 3.3 A function \(f : (X, \tau, I) \to (Y, \sigma)\) is called fuzzy \(\gamma\)-I-continuous (resp. fuzzy \(*\)-I-continuous, fuzzy I-continuous [11], fuzzy pre-I-continuous [10], fuzzy semi-I-continuous [6], fuzzy \(\alpha\)-I-continuous [13] and fuzzy \(\delta\)-I-continuous) if the inverse image of each fuzzy open set of \(Y\) is fuzzy \(\gamma\)-I-open (resp. fuzzy \(*\)-I-dense in itself, fuzzy I-open, fuzzy pre-I-open, fuzzy semi-I-open, fuzzy \(\alpha\)-I-open and fuzzy \(\delta\)-I-open).

Theorem 3.1 Let \(f : (X, \tau, I) \to (Y, \sigma)\) be a function, then the following statements are equivalent:

1. \(f\) is fuzzy \(\gamma\)-I-continuous;
2. For each fuzzy point \( x_\lambda \in X \) and each fuzzy open set \( V \subseteq Y \) containing \( f(x_\lambda) \), there exists \( W \in \text{FGIO}(X) \) such that \( x_\lambda \in W, f(W) \subseteq V \);

3. The inverse image of each fuzzy closed set in \( Y \) is fuzzy \( \gamma \)-I-closed.

**Proof.** (1)⇒(2) Let \( x_\lambda \in X \) and \( V \) be any fuzzy open set of \( Y \) containing \( f(x_\lambda) \). set \( W = f^{-1}(V) \), then by Definition 3.3, \( W \) is a fuzzy \( \gamma \)-I-open containing \( x_\lambda \) and \( f(W) \subseteq V \).

(2)⇒(3) Let \( F \) be a fuzzy closed set in \( Y \). Set \( V = 1 - F \), then \( V \) is fuzzy open in \( Y \). Let \( x_\lambda \in f^{-1}(V) \). By (2), there exists a fuzzy \( \gamma \)-I-open set \( W \) of \( X \) containing \( x_\lambda \) such that \( f(W) \subseteq V \). Thus, we obtain \( x_\lambda \in W \subseteq Cl^*(\text{Int}(W)) \vee \text{Int}(Cl^*(W)) \subseteq Cl^*(\text{Int}(f^{-1}(V))) \vee \text{Int}(Cl^*(f^{-1}(V))) \) and hence \( f^{-1}(V) \subseteq Cl^*(\text{Int}(f^{-1}(V))) \vee \text{Int}(Cl^*(f^{-1}(V))) \). This shows that \( f^{-1}(V) \) is fuzzy \( \gamma \)-I-open in \( X \). Hence \( f^{-1}(F) = 1 - f^{-1}(V) \) is fuzzy \( \gamma \)-I-closed in \( X \).

(3)⇒(1) It is obvious.

**Remark 3.1** Fuzzy \( \gamma \)-I-open set and fuzzy \( \delta \)-I-open set are independent notions as shown with the Example 2.2 in which, if we take \( I = \{0\} \), fuzzy subset \( D \) is fuzzy \( \gamma \)-I-open set but not fuzzy \( \delta \)-I-open and if we take \( I = \text{Power set} \) \( \mathcal{P}(X) \) of \( X \), fuzzy subset \( D \) is fuzzy \( \delta \)-I-open but not fuzzy \( \gamma \)-I-open.

**Theorem 3.2** A fuzzy subset \( A \) of \((X, \tau, I)\) is fuzzy semi-I-open if and only if it is both fuzzy \( \gamma \)-I-open and fuzzy \( \delta \)-I-open.

**Proof.** Let \( A \) be a fuzzy semi-I-open set of \( X \). Then, clearly \( A \) is both fuzzy \( \gamma \)-I-open set and fuzzy \( \delta \)-I-open set by Remark 2.1.

Conversely, let \( A \) be both fuzzy \( \gamma \)-I-open set and fuzzy \( \delta \)-I-open set. Then, we have \( A \subseteq Cl^*(\text{Int}(A)) \vee \text{Int}(Cl^*(A)) \), since \( A \) is fuzzy \( \delta \)-I-open set, therefore \( A \subseteq Cl^*(\text{Int}(A)) \).

**Theorem 3.3** For a fuzzy subset \( A \) of \((X, \tau, I)\), if the condition \((\text{Int}(A))^* \leq \text{Int}(A^*)\) holds, then the following are equivalent:

1. \( A \) is fuzzy \( I \)-open,

2. \( A \) is fuzzy \( \gamma \)-I-open and fuzzy \( *-I \)-dense in itself.

**Proof.** The proof is easy and therefore omitted.

**Theorem 3.4** Let \( f : (X, \tau, I) \to (Y, \sigma) \) be fuzzy \( \gamma \)-I-continuous and \( U \subseteq \tau \). Then the Restriction \( f|_U : (U, \tau|_U, I|_U) \to (Y, \sigma) \) is fuzzy \( \gamma \)-I-continuous.

**Proof** Let \( V \) be any fuzzy open set of \((Y, \sigma)\). Since \( f \) is fuzzy \( \gamma \)-I-continuous, \( f^{-1}(V) \) is \( \gamma \)-I-open set. On the other hand, we have \( f|_U(V) = f^{-1}(V) \cap U \) is fuzzy \( \gamma \)-I-continuous in \((U, \tau|_U, I|_U)\). This shows that \( f|_U : (U, \tau|_U, I|_U) \to (Y, \sigma) \) is fuzzy \( \gamma \)-I-continuous.
Theorem 3.5 Let $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ and $g : (Y, \sigma, J) \rightarrow (Z, \eta)$ be mappings where $X$, $Y$ and $Z$ are fuzzy ideal topological spaces. If $f$ is fuzzy $\gamma$-$I$-continuous and $g$ is fuzzy continuous, then $gof$ is fuzzy $\gamma$-$I$-continuous.

Proof. It is easy and therefore omitted.

Theorem 3.6 A function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is fuzzy semi-$I$-continuous if and only if it is both fuzzy $\gamma$-$I$-continuous and fuzzy $\delta$-$I$-continuous.

Proof. This is an immediate consequence of Theorem 3.2.

Theorem 3.7 If $(\text{Int}(A))^* \leq \text{Int}(A^*)$ holds for each fuzzy subset $A$ of $X$, function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is fuzzy $I$-continuous if and only if it is both fuzzy $\gamma$-$I$-continuous and fuzzy $*)$-$I$-continuous.

Proof. This is an immediate consequence of Theorem 3.3.

4 Fuzzy $\gamma$-$I$-open functions

Definition 4.1 A function $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is called fuzzy $\gamma$-$I$-open (resp. fuzzy pre-$I$-open [13], fuzzy semi-$I$-open [13], fuzzy $\alpha$-$I$-open [13] and fuzzy $I$-open) if the image of each fuzzy open set of $X$ is a fuzzy $\gamma$-$I$-open (resp. fuzzy pre-$I$-open, fuzzy semi-$I$-open, fuzzy $\alpha$-$I$-open and fuzzy $I$-open) set of $Y$.

Remark 4.1 By Definition 4.1 and Remark 2.1, we obtain the following diagram:

\[
\begin{align*}
\text{fuzzy open} & \implies \text{fuzzy } \alpha \text{-} I \text{-open} \implies \text{fuzzy semi-} I \text{-open} \\
\Downarrow & \quad \Downarrow \\
\text{fuzzy } I \text{-open} & \implies \text{fuzzy pre-} I \text{-open} \implies \text{fuzzy } \gamma \text{-} I \text{-open}
\end{align*}
\]

Theorem 4.1 A function $f : (X, \tau, I) \rightarrow (Y, \sigma, I)$ is fuzzy semi-$I$-open if and only if it is both fuzzy $\gamma$-$I$-open and fuzzy $\delta$-$I$-open.

Proof. This is an immediate consequence of Theorem 3.2.

Theorem 4.2 A function $f : (X, \tau, I) \rightarrow (Y, \sigma, I)$ is fuzzy $\gamma$-$I$-open if and only if for each $W \leq Y$ and each fuzzy closed set $F$ of $X$ containing $f^{-1}(W)$, there exists a fuzzy $\gamma$-$I$-closed set $H \leq Y$ containing $W$ such that $f^{-1}(H) \leq F$. 
Proof. Let \( H = 1 - f(1 - F) \). Since \( f^{-1}(W) \leq F \), we have \( f(1 - F) \leq 1 - W \). Since \( f \) is fuzzy \( \gamma \)-I-open, then \( H \) is fuzzy \( \gamma \)-I-closed and \( f^{-1}(H) = 1 - f^{-1}(f(1 - F)) \leq 1 - (1 - F) = F \).

Conversely, let \( U \) be any fuzzy open set of \( X \) and \( W = 1 - f(U) \). Then \( f^{-1}(W) = 1 - f^{-1}(f(U)) \leq 1 - U \) and \( 1 - U \) is fuzzy closed. By the hypothesis, there exists a fuzzy \( \gamma \)-I-closed \( H \) of \( Y \) containing \( W \) such that \( f^{-1}(H) \leq 1 - U \). Then, we have \( H \leq 1 - f(U) \). Therefore, we obtain \( 1 - f(U) \leq H \leq 1 - f(U) \) and \( f(U) \) is fuzzy \( \gamma \)-I-open in \( Y \). This shows that \( f \) is fuzzy \( \gamma \)-I-open.

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References


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