Approximating Fixed Points of Nonexpansive Mappings in CAT(0) Spaces

Thanomsak Laokul

Department of Mathematics, Faculty of Science
Chaing Mai University, Chiang Mai 50200, Thailand
thanom_kul@hotmail.com

Bancha Panyanak

Department of Mathematics, Faculty of Science
Chaing Mai University, Chiang Mai 50200, Thailand
banchap@chiangmai.ac.th

Abstract

Let $C$ be a nonempty closed convex subset of a complete CAT(0) space and $T : C \to C$ be a nonexpansive mapping with $F(T) := \{x \in C : Tx = x\} \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in C$, $x_{n+1} = t_nT[s_nTx_n \oplus (1 - s_n)x_n] \oplus (1 - t_n)x_n$ for all $n \geq 1$,

where $\{t_n\}$ and $\{s_n\}$ are real sequences in $[0,1]$ such that one of the following two conditions is satisfied:

(i) $t_n \in [a,b]$ and $s_n \in [0,b]$ for some $a, b$ with $0 < a \leq b < 1$,
(ii) $t_n \in [a,1]$ and $s_n \in [a,b]$ for some $a, b$ with $0 < a \leq b < 1$.

Then the sequence $\{x_n\}$ $\Delta$–converges to a fixed point of $T$. This is an analog of a result on weak convergence theorem in Banach spaces of Takahashi and Kim [W. Takahashi and G. E. Kim, Approximating fixed points of nonexpansive mappings in Banach spaces, Math. Japonica. 48 no. 1 (1998), 1-9]. Strong convergence of the iterative sequence $\{x_n\}$ is also discussed.

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\footnote{1Corresponding author.}
1 Introduction

A metric space $X$ is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in $X$ is at least as ‘thin’ as its comparison triangle in the Euclidean plane. It is well-known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, $R$–trees (see [1]), Euclidean buildings (see [2]), the complex Hilbert ball with a hyperbolic metric (see [12]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry (see Bridson and Haefliger [1]). Burago, et al. [4] contains a somewhat more elementary treatment, and Gromov [13] a deeper study.

Fixed point theory in a CAT(0) space was first studied by Kirk (see [15] and [16]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and many of papers have appeared (see e. g., [5, 6, 7, 8, 9, 10, 11, 14, 17, 18, 20, 22, 23]).

Recently, Kirk and Panyanak [17] used the concept of $\Delta$–convergence introduced by Lim [19] to prove the CAT(0) space analogs of some Banach space results which involve weak convergence and Dhomponsa and Panyanak [10] obtained $\Delta$–convergence theorems for the Picard, Mann and Ishikawa iterations in the CAT(0) space setting.

The purpose of this paper is to study the iterative scheme defined as follows:

Let $C$ be a nonempty closed convex subset of a complete CAT(0) space and $T : C \to C$ be a nonexpansive mapping with $F(T) := \{ x \in C : Tx = x \} \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in C$,

$$x_{n+1} = t_n T[s_n Tx_n \oplus (1 - s_n)x_n] \oplus (1 - t_n)x_n \quad \text{for all } n \geq 1,$$

where $\{t_n\}$ and $\{s_n\}$ are chosen so that $t_n \in [a, b]$ and $s_n \in [0, b]$ or $t_n \in [a, 1]$ and $s_n \in [a, b]$ for some $a, b$ with $0 < a \leq b < 1$. We show that the sequence $\{x_n\}$ defined by (1) $\Delta$–converges to a fixed point of $T$. This is an analog of a result on weak convergence theorem in Banach spaces of Takahashi and Kim [24]. It is worth mentioning that our result immediately apply to any CAT($\kappa$) space with $\kappa \leq 0$ since any CAT($\kappa$) space is a CAT($\kappa'$) space for every $\kappa' \geq \kappa$ (see [1], p. 165).

2 Preliminary Notes

Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0, l] \subset R$
to $X$ such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image $\alpha$ of $c$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique this geodesic segment is denoted by $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if $Y$ includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consists of three points $x_1, x_2, x_3$ in $X$ (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $E^2$ such that $d_{E^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

\textbf{CAT(0)}: Let $\triangle$ be a geodesic triangle in $X$ and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{E^2}(\bar{x}, \bar{y}).$$

If $x, y_1, y_2$ are points in a CAT(0) space and if $y_0$ is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$ (CN)

This is the (CN) inequality of Bruhat and Tits [3]. In fact (cf. [1], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space $X$. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [9] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

We now give the definition of $\Delta-$convergence.
Definition 2.1 ([17, 19]) A sequence \( \{x_n\} \) in a CAT(0) space \( X \) is said to \( \Delta \)-converge to \( x \in X \) if \( x \) is the unique asymptotic center of \( \{u_n\} \) for every subsequence \( \{u_n\} \) of \( \{x_n\} \). In this case we write \( \Delta - \lim_n x_n = x \) and call \( x \) the \( \Delta \)-limit of \( \{x_n\} \).

Definition 2.2 Let \( C \) be a nonempty subset of a CAT(0) space \( X \) and \( T : C \to X \) be a mapping. \( T \) is called nonexpansive if for each \( x, y \in C \),

\[
d(Tx, Ty) \leq d(x, y).
\]

A point \( x \in C \) is called a fixed point of \( T \) if \( x = Tx \). We denote with \( F(T) \) the set of fixed points of \( T \).

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results.

Lemma 2.3 ([17]) Every bounded sequence in a complete CAT(0) space always has a \( \Delta \)-convergent subsequence.

Lemma 2.4 (Proposition 2.1 of [8]) If \( C \) is a closed convex subset of a complete CAT(0) space and if \( \{x_n\} \) is a bounded sequence in \( C \), then the asymptotic center of \( \{x_n\} \) is in \( C \).

Lemma 2.5 (Proposition 3.7 of [17]) Let \( C \) be a closed convex subset of a complete CAT(0) space \( X \), and let \( T : C \to X \) be a nonexpansive mapping. Then the conditions \( \{x_n\} \Delta \)-converges to \( x \) and \( d(x_n, Tx_n) \to 0 \), imply \( x \in C \) and \( Tx = x \).

Lemma 2.6 Let \((X, d)\) be a CAT(0) space.

(i) [10, Lemma 2.1(iv)] For \( x, y \in X \) and \( t \in [0, 1] \), there exists a unique point \( z \in [x, y] \) such that

\[
d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y).
\]

We use the notation \((1 - t)x \oplus ty\) for the unique point \( z \) satisfying (2).

(ii) [10, Lemma 2.4] For \( x, y, z \in X \) and \( t \in [0, 1] \), we have

\[
d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z).
\]

The following result is a consequence of Lemma 2.9 in [18].

Lemma 2.7 Let \( X \) be a complete CAT(0) space and let \( x \in X \). Suppose \( \{t_n\} \) is a sequence in \([b, c]\) for some \( b, c \in (0, 1) \) and \( \{x_n\}, \{y_n\} \) are sequences in \( X \) such that \( \limsup_n d(x_n, x) \leq r \), \( \limsup_n d(y_n, x) \leq r \), and \( \lim_n d((1 - t_n)x_n \oplus t_ny_n, x) = r \) for some \( r \geq 0 \). Then

\[
\lim_{n \to \infty} d(x_n, y_n) = 0.
\]
3 Main Results

In this section, we prove our main results.

**Theorem 3.1** Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and let $T : C \to C$ be a nonexpansive mapping. Let $\{t_n\}$ and $\{s_n\}$ be sequences in $[0, 1]$. Suppose $x_1 \in C$, and $\{x_n\}$ is defined by

$$x_{n+1} = t_nT[s_nTx_n \oplus (1 - s_n)x_n] \oplus (1 - t_n)x_n$$

for all $n \geq 1$. Then $\lim_{n \to \infty} d(x_n, x^*)$ exists for all $x^* \in F(T)$.

**Proof.** By Lemma 2.6 (ii) and the nonexpansiveness of $T$, for each $x^* \in F(T)$ we have

$$d(x_{n+1}, x^*) = d(t_nT[s_nTx_n \oplus (1 - s_n)x_n] \oplus (1 - t_n)x_n, x^*)$$

$$\leq t_n d(T[s_nTx_n \oplus (1 - s_n)x_n], x^*) + (1 - t_n)d(x_n, x^*)$$

$$\leq t_n d(s_nTx_n \oplus (1 - s_n)x_n, x^*) + (1 - t_n)d(x_n, x^*)$$

$$\leq t_n (s_n d(Tx_n, x^*) + (1 - s_n)d(x_n, x^*)) + (1 - t_n)d(x_n, x^*)$$

$$\leq d(x_n, x^*).$$

Consequently, we have $d(x_n, x^*) \leq d(x_1, x^*)$ for all $n \geq 1$. This implies that $\{d(x_n, x^*)\}_{n=1}^{\infty}$ is bounded and decreasing. Hence $\lim_{n \to \infty} d(x_n, x^*)$ exists. ■

**Theorem 3.2** Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and let $T : C \to C$ be a nonexpansive mapping. Let $\{t_n\}$ and $\{s_n\}$ be sequences in $[0, 1]$ such that $t_n \in [a, b]$ and $s_n \in [0, b]$ or $t_n \in [a, 1]$ and $s_n \in [a, b]$ for some $a, b$ with $0 < a \leq b < 1$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (3). Then $F(T)$ is nonempty if and only if $\{x_n\}$ is bounded and $\lim_{n \to \infty} d(Tx_n, x_n) = 0$.

**Proof.** Suppose that $F(T)$ is nonempty and let $x^* \in F(T)$. Then by Lemma 3.1, $\lim_{n \to \infty} d(x_n, x^*)$ exists and $\{x_n\}$ is bounded. Put

$$c = \lim_{n \to \infty} d(x_n, x^*)$$

and set $y_n = s_nTx_n \oplus (1 - s_n)x_n$ for all $n \geq 1$. Since

$$d(Ty_n, x^*) \leq d(y_n, x^*)$$

$$= d(s_nTx_n \oplus (1 - s_n)x_n, x^*)$$

$$\leq s_n d(Tx_n, x^*) + (1 - s_n)d(x_n, x^*)$$

$$\leq s_n d(x_n, x^*) + (1 - s_n)d(x_n, x^*)$$

$$= (s_n + 1 - s_n)d(x_n, x^*)$$

$$= d(x_n, x^*),$$
we have
\[ \limsup_{n \to \infty} d(Ty_n, x^*) \leq \limsup_{n \to \infty} d(y_n, x^*) \leq c. \tag{5} \]

Further, we have
\[ \lim_{n \to \infty} d(t_nTy_n \oplus (1-t_n)x_n, x^*) = \lim_{n \to \infty} d(x_{n+1}, x^*) = c. \tag{6} \]

Case 1. If \( 0 < a \leq t_n \leq b < 1 \) and \( 0 \leq s_n \leq b < 1 \).

By (4), (5), (6) and Lemma 2.7, we have \( \lim_{n \to \infty} d(Ty_n, x_n) = 0 \).

Case 2. If \( 0 < a \leq t_n \leq 1 \) and \( 0 < a \leq s_n \leq b < 1 \).

By the nonexpansiveness of \( T \), we have \( d(Tx_n, x^*) \leq d(x_n, x^*) \) for all \( n \geq 1 \).

This implies
\[ \limsup_{n \to \infty} d(Tx_n, x^*) \leq c. \tag{7} \]

Now,
\[ d(x_{n+1}, x^*) \leq t_n d(Ty_n, x^*) + (1-t_n)d(x_n, x^*) \]
\[ \leq t_n d(y_n, x^*) + (1-t_n)d(x_n, x^*) \]
\[ = t_n d(y_n, x^*) + d(x_n, x^*) - t_n d(x_n, x^*). \]

This implies
\[ \frac{d(x_{n+1}, x^*) - d(x_n, x^*)}{t_n} \leq d(y_n, x^*) - d(x_n, x^*). \]

Thus
\[ d(x_{n+1}, x^*) - d(x_n, x^*) \leq \frac{d(x_{n+1}, x^*) - d(x_n, x^*)}{t_n} \leq d(y_n, x^*) - d(x_n, x^*). \]

Combining this inequality and (5), we have
\[ c \leq \liminf_{n \to \infty} d(y_n, x^*) \leq \limsup_{n \to \infty} d(y_n, x^*) \leq c. \]
Therefore
\[ c = \lim_{n \to \infty} d(y_n, x^*) = \lim_{n \to \infty} d(s_n Tx_n \oplus (1 - s_n)x_n, x^*). \tag{8} \]

By (4), (7), (8) and Lemma 2.7, we have \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \).

Conversely, suppose that \( \{x_n\} \) is bounded and \( \lim_{n \to \infty} d(Tx_n, x_n) = 0 \).

Let \( A(\{x_n\}) = \{x\} \). Then \( x \in C \) by Lemma 2.4. Since \( d(x_n, Tx) \leq d(x_n, Tx_n) + d(Tx_n, Tx) \) for all \( n \geq 1 \), then
\[ \limsup_{n \to \infty} d(x_n, Tx) \leq \limsup_{n \to \infty} d(x_n, x). \]

By the unique of asymptotic centers, we have \( Tx = x \). Therefore, \( x \) is a fixed point of \( T \).

The following theorem is an analog of Theorem 1 of [24].

**Theorem 3.3** Let \( C \) be a nonempty closed convex subset of a complete \( CAT(0) \) space \( X \), and let \( T : C \to C \) be a nonexpansive mapping with \( F(T) \neq \emptyset \). Let \( \{t_n\} \) and \( \{s_n\} \) be sequences in \([0, 1]\) so that \( t_n \in [a, b] \) and \( s_n \in [0, b] \) or \( t_n \in [a, 1] \) and \( s_n \in [a, b] \) for some \( a, b \) with \( 0 < a \leq b < 1 \). From arbitrary \( x_1 \in C \), define the sequence \( \{x_n\} \) by the recursion (3). Then \( \{x_n\} \) \( \Delta \)-converges to a fixed point of \( T \).

**Proof.** Theorem 3.2 guarantees that \( \{x_n\} \) is bounded and \( \lim_{n \to \infty} d(Tx_n, x_n) = 0 \).

We now let \( \omega_w(x_n) := \bigcup A(\{u_n\}) \) where the union is taken over all sub-sequences \( \{u_n\} \) of \( \{x_n\} \). We claim that \( \omega_w(x_n) \subset F(T) \). Let \( u \in \omega_w(x_n) \), then there exists a subsequence \( \{u_n\} \) of \( \{x_n\} \) such that \( A(\{u_n\}) = \{u\} \).

By Lemma 2.3 and 2.4 there exists a subsequence \( \{v_n\} \) of \( \{u_n\} \) such that \( \Delta - \lim_n v_n = v \in C \). Since \( \lim_n d(v_n, T v_n) = 0 \), then \( v \in F(T) \) by Lemma 2.5.

We claim that \( u = v \). Suppose not, since \( T \) is nonexpansive and \( v \in F(T) \), \( \lim_n d(x_n, v) \) exists by Theorem 3.1. Then by the uniqueness of asymptotic centers,
\[
\limsup_n d(v_n, v) < \limsup_n d(v_n, u)
\leq \limsup_n d(u_n, u)
\leq \limsup_n d(u_n, v)
= \limsup_n d(x_n, v)
= \limsup_n d(v_n, v)
\]
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a contradiction, and hence \(u = v \in F(T)\). To show that \(\{x_n\}\) \(\Delta\)–converges to a fixed point of \(T\), it suffices to show that \(\omega_w(x_n)\) consists of exactly one point. Let \(\{u_n\}\) be a subsequence of \(\{x_n\}\). By Lemmas 2.3 and 2.4 there exists a subsequence \(\{v_n\}\) of \(\{u_n\}\) such that \(\Delta \lim_n v_n = v \in C\). Let \(A(\{u_n\}) = \{u\}\) and \(A(\{x_n\}) = \{x\}\). We have seen that \(u = v\) and \(v \in F(T)\). We can complete the proof by showing that \(x = v\). Suppose not, since \(\{d(x_n, v)\}\) is convergent,

\[
\limsup_n d(v_n, v) < \limsup_n d(v_n, x) \\
\leq \limsup_n d(x_n, x) \\
< \limsup_n d(x_n, v) \\
= \limsup_n d(v_n, v)
\]
a contradiction, and hence the conclusion follows.

Another result in [24], the authors prove that the sequence \(\{x_n\}\) defined by (3) converges strongly to a fixed point of a nonexpansive mapping \(T\) whose domain is a nonempty closed convex subset \(C\) of a strictly convex Banach space \(E\) and its image \(T(C)\) is contained in a compact subset of \(C\) (see [24, Theorem 3]). The main tool in proving the result is Mazur’s theorem [21] which is stated that “The closed convex hull of a compact subset of a Banach space is itself compact”. An interesting question is:

**Question 1.** Can Theorem 3 of [24] extended to CAT(0) spaces? equivalently, if \(C\) is a nonempty closed convex subset of a complete CAT(0) space \(X\), and if \(T : C \to C\) is a nonexpansive mapping such that \(T(C)\) is contained in a compact subset of \(C\), \(x_1 \in C\) and \(\{x_n\}\) is given by

\[
x_{n+1} = t_n T[s_n T x_n \oplus (1 - s_n) x_n] \oplus (1 - t_n) x_n
\]

for all \(n \geq 1\), where \(\{t_n\}\) and \(\{s_n\}\) be sequences in \([0, 1]\) such that \(t_n \in [a, b]\) and \(s_n \in [0, b]\) or \(t_n \in [a, 1]\) and \(s_n \in [a, b]\) for some \(a, b\) with \(0 < a \leq b < 1\), does the sequence \(\{x_n\}\) converge strongly to a fixed point of \(T\)?

One way to solve Question 1 is to prove Mazur’s theorem in CAT(0) spaces. Thus, we should state:

**Question 2.** Is the closed convex hull of a compact subset of a complete CAT(0) space compact?

However, for Question 1, if the assumption “\(T(C)\) is contained in a compact subset of \(C\)” is replaced by “\(C\) is a compact subset of \(X\)”, then we can prove the strong convergence of \(\{x_n\}\) as the following result.
Theorem 3.4 Let $C$ be a nonempty compact convex subset of a complete CAT(0) space $X$, and let $T : C \to C$ be a nonexpansive mapping. Let $\{t_n\}$ and $\{s_n\}$ be sequences in $[0, 1]$ such that $t_n \in [a, b]$ and $s_n \in [0, b]$ or $t_n \in [a, 1]$ and $s_n \in [a, b]$ for some $a, b$ with $0 < a \leq b < 1$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (9). Then $\{x_n\}$ converges strongly to a fixed point of $T$.

Proof. We first note that $F(T)$ is nonempty by [16, Theorem 12]. By the compactness of $C$, we see that $\{x_n\}$ has a strongly convergent subsequence $\{x_{n_k}\}$ whose limit we shall denote by $z$. Then, by Theorem 3.2 and the nonexpansiveness of $T$,

$$d(z, Tz) \leq d(z, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Tz) \leq 2d(z, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) \to 0 \text{ as } k \to \infty.$$ 

Therefore $z \in F(T)$. By Theorem 3.1 $\lim_n d(x_n, z)$ exists, thus $z$ is the strong limit of the sequence $\{x_n\}$ itself. ■

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