Approximation of Conjugate of Functions

Belonging to the Generalized Lipschitz Class by Lower Triangular Matrix Means

Shyam Lal

Department of Mathematics, Banaras Hindu University
Varanasi -221005 (India)
shyam_lal@rediffmail.com

Jitendra Kumar Kushwaha

Department of Mathematics, Banaras Hindu University
Varanasi -221005 (India)
k.jitendrakumar@yahoo.com

Abstract

In this paper, two new theorems on the degree of approximation of $\tilde{f}(x)$, conjugate of function $f \in \text{Lip}_\alpha$ and $f \in \text{Lip}\{\tilde{\xi}(t), p\}$ class, by lower triangular matrix means of conjugate series of the Fourier series have been established.

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1. Introduction and Definitions

The degree of approximation of a function $f \in \text{Lip} \alpha$ and $f \in \text{Lip}(\tilde{\xi}(t), p)$ has been determined by several investigators like Alexits [1], Sahney and Goel [12], Chandra ([4],[5]), Qureshi ([8], [9]) and Leindler [6], by Cesaro means and Nörlund means of Fourier series. Working in the same direction Qureshi ([10],[11]) have determined the degree of approximation of $\tilde{f}(x)$, conjugate of a function $f \in \text{Lip} \alpha$ and $f \in \text{Lip}(\tilde{\xi}(t), p)$ by Nörlund means of conjugate series of a Fourier series. But till now nothing seems to have been done so for to determined the degree of approximation of $\tilde{f}(x)$, conjugate of a function $f \in \text{Lip} \alpha$ and $f \in \text{Lip}(\tilde{\xi}(t), p)$, by lower triangular matrix means of conjugate series of the Fourier series. Lower triangular matrix summability transformation includes (C, 1), (C, $\delta$), (N, $p_n$) and (N, p, q) methods as particular cases. The generalized Lipschitz class $\text{Lip}(\tilde{\xi}(t), p)$ is a generalization of $\alpha$ and $\text{Lip}(\alpha, p)$. The purpose of this paper is to determine the degree of approximation of $\tilde{f}(x)$, conjugate of function $f \in \text{Lip} \alpha , 0 < \alpha \leq 1$ and $f \in \text{Lip}(\tilde{\xi}(t), p)$, by lower triangular matrix means.

A function $f \in \text{Lip} \alpha$ if $\left| f(x + t) - f(x - t) \right| = O(t^\alpha)$, $0 < \alpha \leq 1$.

A function $f(x) \in \text{Lip}(\alpha, p)$ for $0 \leq x \leq 2\pi$, if

$$\left( \int_0^{2\pi} |f(x + t) - f(x)|^p \, dx \right)^{1/p} = O(\left| t \right|^{\alpha}) , \quad 0 < \alpha \leq 1 , p \geq 1 \quad (\text{McFadden [7]}) .$$

Given a positive increasing function $\tilde{\xi}(t)$ of $t$ and an integer $p \geq 1$,

$f(x) \in \text{Lip}(\tilde{\xi}(t), p)$ if $\left( \int_0^{2\pi} |f(x + t) - f(t)|^p \, dx \right)^{1/p} = O(\tilde{\xi}(t))$ \quad (Siddiqi [13]).

In case $\tilde{\xi}(t) = t^\alpha$ then $\text{Lip}(\tilde{\xi}(t), p)$ coincides to $\text{Lip}(\alpha, p)$.

If $p \to \infty$ in $\text{Lip}(\alpha, p)$ then it coincides to $\text{Lip} \alpha$.

$L_\infty$ – norm of a function $f : R \to R$ is defined by $\|f\|_\infty = \sup \{|f(x) : x \in R|\}$

$L_p$ – norm is defined by $\|f\|_p = \left( \int_0^{2\pi} |f(x)|^p \, dx \right)^{1/p} , p \geq 1$.

The degree of approximation $E_n(f)$ of a function $f : R \to R$ by a trigonometric polynomial $t_n$ of order $n$ is defined by $\text{(Zygmund [16], p. 114)}$

$$\|t_n - f\|_p = \min \|t_n - f\|_p .$$
Let \( f \) be periodic with period \( 2\pi \) and integrable over \((-\pi, \pi)\) in Lebesgue sense and \( f(x) \in \text{Lip}(\xi(t), p) \). Let its Fourier series be given by

\[
f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n(x). \tag{1}
\]

The conjugate series of the Fourier series (1) is given by

\[
\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) = -\sum_{n=1}^{\infty} B_n(x). \tag{2}
\]

If \( f \) is Lebesgue integrable then

\[
\tilde{f}(x) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(t) \cot(t/2) \, dt = -\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \psi(t) \cot(t/2) \, dt
\]

exists for almost all \( x \) (Zygmund [16], p.131). \( \tilde{f}(x) \) is called the conjugate function of \( f(x) \).

Let \( T = \{a_{n,k}\} \) be an infinite lower triangular matrix satisfying Töeplitz [15] conditions of regularity, i.e. \( \sum_{k=0}^{n} a_{n,k} \to 1 \), as \( n \to \infty \), \( a_{n,k} \to 0 \), for \( k > n \) and \( \sum_{k=0}^{n} |a_{n,k}| \leq M \) a finite constant.

Let \( \sum_{n=0}^{\infty} u_n \) be an infinite series whose \( k^{th} \) partial sum \( s_k = \sum_{n=0}^{k} u_n \). The sequence to sequence transformation \( t_n = \sum_{k=0}^{n} a_{n,k} s_k \) defines the sequence \( \{t_n\} \) of lower triangular matrix summability means of sequence \( \{s_n\} \) generated by the sequence of coefficients \( \{a_{n,k}\} \). The series \( \sum_{n=0}^{\infty} u_n \) is said to be summable to sum \( s \) by lower triangular matrix method if \( \lim_{n \to \infty} t_n \) exists and is equal to \( s \) (Zygmund [16], p.74) and we write \( t_n \to s(T) \), as \( n \to \infty \).

We use the following notations:

\[
\psi(t) = f(x + t) - f(x - t), \quad A_{n,\tau} = \sum_{k=n-\tau}^{n} a_{n,k},
\]
\[ \tau = \left\lfloor \frac{1}{t} \right\rfloor = \text{The greatest integer not greater than } (1/t) \]

\[ M_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \cos(k + 1/2)t \sin(t/2) . \]

2. Theorems

We prove the following theorems:

**Theorem 1**: Let \( T = (a_{n,k}) \) be an infinite regular lower triangular matrix such that the element \( (a_{n,k}) \) be non-negative, non-decreasing with \( k \leq n \). If \( f : \mathbb{R} \to \mathbb{R} \) is \( 2\pi \)-periodic, Lebesgue integrable on \((-\pi, \pi)\) and \( f \in \text{Lip}_\alpha , 0 < \alpha \leq 1 \), then the degree of approximation of its conjugate function \( \tilde{f} \) by lower triangular matrix means \( \tilde{t}_n(x) = \sum_{k=0}^{n} a_{n,k} \tilde{s}_k(x) \) of conjugate series of Fourier series (2) satisfies, \( n=0,1,2 \ldots \),

\[
\left\| \tilde{t}_n - f \right\|_{p} = \begin{cases} 
O((n + 1)^{-\alpha}) , & 0 < \alpha < 1 \\
O(\log(n + 1)\pi e/(n + 1)) , & \alpha = 1 .
\end{cases}
\]

**Theorem 2.** The degree of approximation of a function \( \tilde{f}(x) \), conjugate of a function \( f \in \text{Lip}(\xi(t),p) \), by the lower triangular matrix means \( \tilde{t}_n(x) \) of conjugate Fourier series (2) is given by

\[
\left\| \tilde{t}_n - \tilde{f} \right\|_{p} = O\left( (n + 1)^{1/p} \xi \left( \frac{1}{n + 1} \right) \right)
\]

provided \( \xi(t) \) is positive increasing function of \( t \) satisfying

\[
\left\{ \int_{0}^{1/(n+1)} \left( \frac{t |\psi(t)|}{\xi(t)} \right)^p dt \right\}^{1/p} = O\left( (n + 1)^{-1} \right) ,
\]

(3)

\[
\left\{ \int_{1/(n+1)}^{\pi} \left( \frac{t^{-\delta} |\psi(t)|}{\xi(t)} \right)^p dt \right\}^{1/p} = O\left( (n + 1)^{\delta} \right)
\]

(4)

where \( \delta \) is an arbitrary number such that \( q(1-\delta)-1 > 0 \), \( q \) the conjugate index of \( p \) and the condition (3) and (4) hold uniformly in \( x \).
3. Lemma

For the proof of our theorems, the following lemma is required.

**Lemma.** Under the condition of our theorem on \((a_{n,k})\),

\[
M_n(t) = O\left(\frac{A_{n,\tau}}{t}\right), \quad \text{for } (n+1)^{-1} < t \leq \pi.
\]

**Proof.** For \((n+1)^{-1} < t \leq \pi, \sin(t/2) \geq (t/\pi), \tau \leq n\), we have

\[
|M_n(t)| = \left|\frac{1}{2\pi} \sum_{k=0}^{n-\tau-1} a_{n,k} \cos(k/2)t + \frac{1}{2\pi} \sum_{k=n-\tau}^{n} a_{n,k} \cos(k/2)t\right|
\]

\[
\leq \frac{1}{2\pi} \left|\sum_{k=0}^{n-\tau-1} a_{n,k} \cos(k/2)t + \sum_{k=n-\tau}^{n} a_{n,k} \cos(k/2)t\right|
\]

\[
\leq \frac{1}{2\pi} \left[2a_{n,n-\tau-1} \max_{0 \leq \tau \leq n-\tau-1} \left|\sum_{k=0}^{\tau} \cos(k/2)t + \sum_{k=n-\tau}^{n} a_{n,k} \cos(k/2)t\right|\right]
\]

\[
= \frac{1}{2\pi} \left[O\left(\frac{a_{n,n-\tau-1}}{t}\right) + A_{n,\tau}\right]
\]

and \(A_{n,\tau} = \sum_{k=n-\tau}^{n} a_{n,k} = a_{n,n-\tau} + a_{n,n-\tau+1} + \ldots + a_{n,n}\)

\[
\geq a_{n,n-\tau-1} + a_{n,n-\tau-1} + \ldots + a_{n,n-\tau-1}
\]

\[
= (\tau+1)a_{n,n-\tau-1}
\]

\[
\geq \left(\frac{a_{n,n-\tau-1}}{t}\right).
\]

Therefore, \(|M_n(t)| = O\left(\frac{A_{n,\tau}}{t}\right)\).

4. Proof of the Theorem 1

The \(k\)th partial sum of the conjugate series of the Fourier series (2) is given by

\[
\tilde{S}_n(x) = -\frac{1}{2\pi} \int_{0}^{\pi} \cot(t/2) \psi(t)dt + \frac{1}{2\pi} \int_{0}^{\pi} \frac{\cos(n+1/2)t}{\sin(t/2)} \psi(t)dt
\]

\[
\tilde{S}_n(x) = \left\{-\frac{1}{2\pi} \int_{0}^{\pi} \cot(t/2) \psi(t)dt\right\} = \frac{1}{2\pi} \int_{0}^{\pi} \frac{\cos(n+1/2)t}{\sin(t/2)} \psi(t)dt
\]
Then
\[ \sum_{k=0}^{n} a_{n,k} \left\{ \tilde{S}_n(x) - \left( -\frac{1}{2\pi} \int_0^{\pi} \cot(t/2) \psi(t) \, dt \right) \right\} = \frac{1}{2\pi} \int_0^{\pi} \left( \sum_{k=0}^{n} a_{n,k} \frac{\cos(n+1/2)t}{\sin(t/2)} \right) \psi(t) \, dt \]

or, \( \tilde{t}_n(x) - f(x) = \int_0^{\pi} \psi(t) M_n(t) \, dt \)
\[ = \int_0^{1/(n+1)} \psi(t) M_n(t) \, dt + \int_{1/(n+1)}^{\pi} \psi(t) M_n(t) \, dt \]
\[ = I_1 + I_2 \quad (5) \]
\[ |f(x + t) - f(x)| = O(t^\alpha), \ f \in \text{Lip}_\alpha. \]
\[ |\psi(t)| = |f(x + t) - f(x - t)| \leq |f(x + t) - f(x)| + |f(x) - (x + t)|. \]
\[ = O(t^\alpha) + O(t^\alpha) = O(t^\alpha) \]

Then \( \psi(t) \in \text{Lip}_\alpha. \)

Now, for \( 0 < t \leq 1/(n+1) \), we have
\[ |I_1| = \left| \int_0^{1/(n+1)} \psi(t) \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,k} \frac{\cos(k+1/2)t}{\sin(t/2)} \, dt \right| \]
\[ \leq \frac{1}{2\pi} \int_0^{1/(n+1)} |\psi(t)| \left| \sum_{k=0}^{n} a_{n,k} \frac{\cos(k+1/2)t}{\sin(t/2)} \right| \, dt \]
\[ = \left( \int_0^{1/(n+1)} t^\alpha O(H(t)) \, dt \right) \]
\[ = O(t^{1/1-(n+1)}). \quad (6) \]

Using lemma, for \( 1/(n+1) < t \leq \pi \), we have
\[ |I_2| = \left| \int_{1/(n+1)}^{\pi} \psi(t) M_n(t) \, dt \right| \]
\[ = \left( \int_{1/(n+1)}^{\pi} t^\alpha O\left( \frac{A_{n,t}}{t} \right) \, dt \right) \]
\[ = O \left( \int_{1/(n+1)}^{\pi} A_{n,y} \, dy \right). \]
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\[ \begin{align*}
&= O \left( \frac{A_{n,n}}{n+1} \right) \int_{1/\pi}^{n+1} \frac{dy}{y^\alpha} + O \left( \frac{A_{n,n/\pi}}{n+1} \right) \int_{1/\pi}^{n+1} \frac{dy}{y^\alpha} \\
&= O \left( \frac{1}{n+1} \right) \int_{1/\pi}^{n+1} \frac{dy}{y^\alpha} \quad \left( \therefore \frac{A_{n,n}}{y^\alpha} \text{ is monotonic} \right) \\
&= O \left( \frac{1}{n+1} \right) \left\{ \frac{1}{1-\alpha} \left[ \frac{1}{(n+1)^{\alpha-1}} - \left( \frac{1}{\pi} \right)^{-\alpha+1} \right] \right\}, \quad 0 < \alpha < 1 \\
&= O((n+1)^{-\alpha}), \quad 0 < \alpha < 1 \\
&= O(\log(n+1)\pi/(n+1)), \quad \alpha = 1.
\end{align*} \]

Combining from (5) to (7), we have

\[ \left\| t_n - \tilde{f} \right\|_{\infty} = \sup_{x} \left\| t_n(x) - \tilde{f}(x) \right\| : x \in \mathbb{R} \]

\[ = \begin{cases} 
O((n+1)^{-\alpha}), & 0 < \alpha < 1 \\
O(\log(n+1)\pi e/(n+1)), & \alpha = 1.
\end{cases} \]

5. Proof of Theorem 2.

Following the proof of theorem 1,

\[ \tilde{t}_n(x) - \tilde{f}(x) = \int_{0}^{1/(n+1)} \psi(t)M_n(t)dt + \int_{1/(n+1)}^{\pi} \psi(t)M_n(t)dt \]

\[ = I_1 + I_2 \quad (8) \]

Using Hölder’s inequality, \( \psi(t) \in \text{Lip} \left( \xi(t), p \right) \), (3), \( \sin (t/2) \geq (t/\pi) \) and second mean value theorem for integrals, we have

\[ \left| I_1 \right| \leq \left\{ \int_{0}^{1/(n+1)} \left( \frac{t|\psi(t)|}{\xi(t)} \right)^p dt \right\}^{1/p} \left\{ \int_{0}^{1/(n+1)} \left( \frac{|\xi(t)|M_n(t)}{t} \right)^q dt \right\}^{1/q} \]

\[ \leq \left\{ \int_{0}^{1/(n+1)} \left( \frac{t|\psi(t)|}{\xi(t)} \right)^p dt \right\}^{1/p} \left\{ \int_{0}^{1/(n+1)} \left( \frac{\xi(t)|\cos(k+1/2)t|}{t|\sin(t/2)|} \right)^q dt \right\}^{1/q} \]

\[ = O \left( \frac{1}{n+1} \right) \left\{ \int_{0}^{1/(n+1)} \left( \frac{\xi(t)}{t^2} \right)^q dt \right\}^{1/q} \]
\[
= O\left(\left(\frac{1}{n+1}\right)^{1/(n+1)}\int_{\epsilon}^{1} \frac{1}{t^{2q}} dt\right)^{1/q}, \text{ for some } 0 < \epsilon < 1/(n+1)
\]
\[
= O\left(\left(\frac{1}{n+1}\right)^{1/(n+1)}\left(\frac{1}{(n+1)^{2q-4} - \epsilon^{1-2q}}\right)^{1/q}\right)
\]
\[
= O\left((n+1)^{1/p} \xi \left(\frac{1}{n+1}\right)\right), \quad \left(\because p^{-1} + q^{-1} = 1\right) \tag{9}
\]

Using lemma, (4) and hypothesis of the theorem, we have
\[
|I_2'| = \left[ \int_{1/(n+1)}^{\pi} \frac{t^{-\delta} \psi(t)}{\xi(t)} \right]^{1/p} \left[ \int_{1/(n+1)}^{\pi} \frac{\xi(t) |M_n(t)|^q}{t^{-\delta}} \right]^{1/q}
\]
\[
= O\left((n+1)^{\delta} \right) \left[ \int_{1/(n+1)}^{\pi} \frac{\xi(t) \left[A_n(t)\right]^q}{t^{-\delta}} \right]^{1/q}
\]
\[
= O\left((n+1)^{\delta} \right) \left[ \int_{1/(n+1)}^{\pi} \left(\frac{\xi(t) A_n(t)}{t^{1-\delta}}\right)^q dt \right]^{1/q}
\]
\[
= O\left((n+1)^{\delta} \right) \left[ \int_{1/(n+1)}^{\pi} \left(\frac{\xi(y) A_n(y)}{y^{1-\delta}}\right)^q \frac{dy}{y^2} \right]^{1/q}
\]
\[
= O\left((n+1)^{\delta} \right) \left[ \int_{1/(n+1)}^{\pi} \frac{dy}{y^{q+\delta}} \right]^{1/q}
\]
\[
= O\left((n+1)^{\delta} \right) \left(\frac{(n+1)^q(1-\delta) - 1}{\pi^q(1-\delta) - 1}\right)^{1/q}
\]
\[
= O\left((n+1)^{\delta} \xi \left(\frac{1}{n+1}\right)\right) O\left((n+1)^{q(1-\delta) - 1}\right)^{1/q}
\]
\[
= O\left((n+1)^{1/p} \xi \left(\frac{1}{n+1}\right)\right) \tag{10}
\]

Combining from (8) to (10), we have
\[
\left\| \tilde{t}_n - f \right\|_p = O \left( (n + 1)^{1/p} \frac{1}{\xi(n + 1)} \right).
\]

6. Corollaries

Following corollaries may be derived from Theorem 2.

**Cor.1.** If \( \xi(t) = t^\alpha \) then the degree of approximation of a function \( \tilde{f}(x) \), conjugate of \( f \in \text{Lip}(\alpha, p), \frac{1}{p} < \alpha < 1 \) by lower triangular matrix means \( t_n(x) \) of the conjugate series of the Fourier series (2) is given by

\[
\left\| \tilde{t}_n - f \right\|_p = O \left( (n + 1)^{-\alpha + \frac{1}{p}} \right).
\]

**Cor.2.** If \( p \to \infty \) in corollary 1, then for \( 0 < \alpha < 1 \),

\[
\left\| \tilde{t}_n - f \right\|_{\infty} = O \left( (n + 1)^{-\alpha} \right).
\]

**Cor.3** If \( a_n,k = \frac{p_n-k}{p_n} \), \( p_n \neq 0 \) and \( \xi(t) = t^\alpha \) then the degree of approximation of \( \tilde{f}(x) \), conjugate of \( f \in \text{Lip}(\alpha, p) \) by Nörlund means \( \tilde{t}_n = \frac{1}{p_n} \sum_{k=0}^{n} s_k \) of the conjugate series of Fourier series is given by

\[
\left\| \tilde{t}_n - f \right\|_p = O \left( (n + 1)^{-\alpha + \frac{1}{p}} \right).
\]

**Cor.4** If \( a_n,k = \frac{p_n-k}{p_n} \) and \( \xi(t) = t^\alpha \) and \( p \to \infty \) then the degree of approximation of \( \tilde{f}(x) \), conjugate of \( f \in \text{Lip}\alpha \) by Nörlund means \( \tilde{t}_n = \frac{1}{p_n} \sum_{k=0}^{n} s_k \) of the conjugate series of Fourier series is given by

\[
\left\| \tilde{t}_n - f \right\|_{\infty} = \begin{cases} O \left( (n + 1)^{-\alpha} \right), & 0 < \alpha < 1 \\ O(\log(n + 1)\pi e/(n + 1)), & \alpha = 1. \end{cases}
\]
If $a_{n,k} = \frac{p_{n+k}}{q_k}$ such that $R_n = \sum_{k=0}^{n} p_{n-k} q_k \neq 0$, $\frac{R(y)}{y^\alpha}$ is monotonic non-decreasing then degree of approximation of $f(x)$, conjugate of a function $f \in \text{Lip}_\alpha$, by generalized Nörlund means $t_n = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k s_k(x)$ of the conjugate series (2) satisfies
\[
\left\| t_n - f \right\|_\infty = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0 < \alpha < 1 \\ O\left((\log(n+1))e/(n+1)\right), & \alpha = 1. \end{cases}
\]

**Remarks:**

**Remark 1.** (1) The degree of approximation $\left\| f - t_n \right\|_p = O\left((n+1)^{-\alpha+\frac{1}{p}}\right)$ determined by Qureshi ([11], p.561, L.12) tends to $\infty$ if $0 < \alpha = \frac{1}{3} < 1$ and $p=2$ and also for other values. Therefore, this deficiency has motivated to investigate degree of approximation of conjugate of functions belonging to $\text{Lip}(\alpha, p)$ considering $\frac{1}{p} < \alpha < 1$.

**Remark 2.** There are several results, for example, Alexits [1], Chandra [4], Sahney & Goel [12], Alexits & Leindler [2] and Bernstein [3] for the degree of approximation of functions $f \in \text{Lip}_\alpha$, but most of these results are not satisfied for $n= 0, 1$ or $\alpha = 1$. Therefore, this deficiency has motivated to investigate degree of approximation of functions belonging to $\text{Lip}_\alpha$ considering cases $0 < \alpha < 1$ and $\alpha = 1$ separately. Considering these specific cases separately, we have obtained better and sharper estimate of $\tilde{f}(x)$, conjugate of $f \in \text{Lip}_\alpha$, than all previously known results in this direction.

**References**


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