Generalization of Linear Shift Invariant System

in the Fractional Domain

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Abstract

Fractional Fourier transform is one of a flourishing field of active research due to its wide range of applications. It is well-known that fractional Fourier transform is linear, but not shift invariant as that of conventional Fourier transform. Linear shift invariant systems can be expressed in terms of convolution of two functions. Convolution for fractional Fourier transform, defined by Almeida is redefined by Zayed A H in order to satisfy the convolution theorem. Akay O had formulated linear fractional shift invariant system through fractional operators.

Purpose of this paper is to define linear fractional shift invariant system in terms of its response to a unit impulse and also to show that fractional type convolution as defined by Zayed, can be used in dealing these linear fractional shift invariant systems as in case of conventional Fourier transform.

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1. Introduction

The generalization of the Fourier transform into fractional domain of time- frequency plane is given by Namias [4]
\[
\{F^\alpha f(t)\}(s) = F^\alpha(s) = \sqrt{\frac{1-i\cot \varphi}{2\pi}} \int_{-\infty}^{\infty} e^{\frac{i(s^2+1)}{2}\cot \varphi - i\varphi s} f(t) \, dt, \quad \text{where} \quad \varphi = \frac{\alpha \pi}{2}. \tag{1}
\]

This transform is linear but not shift invariant. Akay [1] had defined unitary fractional shift operator
\[
(R^\varphi_s)(t) = s(t - \rho \cos \varphi) \exp(-i\pi \rho^2 \cos \varphi \sin \varphi + 2\pi i t \rho \sin \varphi),
\]
which shifts the support of the signal \( s(t) \) radially in the time-frequency plane by an amount equal to \( \rho \) along the radial orientation \( \varphi \), measured anticlockwise from the time axis.

Linearity and time invariance are basic properties of system. Many physical processes can be modeled by the linear time-invariant systems which can be analyzed in great details. Moreover the complete characterization of linear shift invariant system can be developed in terms of its response to a unit impulse.

As mentioned in [5], one of the most important properties of the Fourier transform with regard to its use in dealing with linear time invariant system, is its effect on convolution operator. It is derived that if \( x(t), y(t) \) are input and output of linear time invariant system with impulse response \( h(t) \) then \( y(t) = h(t) \ast x(t) \), where
* denotes convolution operator given by \( y(t) = \int_{-\infty}^{\infty} x(T) h(T-t) \, dT \),
(2)
then the convolution theorem of the Fourier transform gives
\[
Y(w) = H(w) X(w), \tag{3}
\]
where \( X(w), Y(w) \) are Fourier transforms of \( x(t) \) and \( y(t) \) respectively. Moreover \( H(w) \) is Fourier transform of impulse response \( \delta(t-\tau) \) i.e. \( H(w) = e^{-i\omega \tau} \).

Thus (3) gives
\[
Y(w) = e^{-i\omega \tau} X(w).
\]
Note that this result is consistent with the shifting property of Fourier transform. That is if
\[
\{F x(t)\}(w) = X(w), \quad \text{then}
\{F x(t-\tau)\}(w) = e^{-i\omega \tau} X(w).
\]

In this paper a new linear fractional invariant system is formulated which will give similar type of consistency with the corresponding shifting property of fractional Fourier transform as shown above for conventional Fourier transform.

It is known that if fractional Fourier transform of \( x(t) \) is denoted by \( \{F^\alpha x(t)\}(s) \) then
\[
\{F^\alpha x(t-\tau)\}(s) = \{F^\alpha x(t)\}(s - \tau \cos \varphi) \exp(i\tau^2 \sin \varphi \cos \varphi - i\tau \sin \varphi).
\]
Simple calculations show that right hand side of above equation can also be expressed as 

\[ \{ F_{\alpha} x(t-\tau) \} (s) = e^{\frac{i}{2} \cot \phi \cdot \frac{x}{\alpha}} \cdot e^{i \frac{\cot \phi \cdot \cos \phi - \sin \phi}{2}} \cdot \{ F_{\alpha} e^{i \cot \phi \cdot x(t)} \} (s) \]

\[ = e^{\frac{i}{2} \cot \phi \cdot \frac{x}{\alpha}} \cdot \{ F_{\alpha} e^{i \cot \phi \cdot x(t)} \} (s) \]

(4)

2. Convolution for the fractional Fourier transform

Almeida [3] had derived convolution of two functions for fractional Fourier transform. But as explained by Zayed [6] this convolution formula did not generalize the classical result (2) of the Fourier transform. So Zayed had given a new definition for fractional type convolution “in terms of usual convolution * given in (2).

\[ h(x) = (f \circ g)(x) = \frac{1 - i \cot \phi}{2\pi} e^{\frac{i}{2} \cot \phi} \cdot (f * g). \]

(5)

Where \( \tilde{f} = e^{\frac{i}{2} \cot \phi} f(x) \). He had also shown that if \( F_{\alpha}^\alpha (u), G_{\alpha}^\alpha (u) \) and denote fractional Fourier transform of \( f, g \) and \( h \) respectively then

\[ H_{\alpha}^\alpha (u) = e^{\frac{i}{2} \cot \phi} F_{\alpha}^\alpha (u) G_{\alpha}^\alpha (u) \]

(6)

which can be considered as generalization of (3) since the coefficient term is of unit magnitude.

3. Linear fractional shift invariant system

We defined linear fractional shift invariant system as fractional convolution (Zayed type) of two functions, one is \( e^{i \cot \phi \cdot x(t)} \), and other is \( h(t) \), where \( h(t) \) is impulse response at \( t \), that is \( y(t) = (e^{i \cot \phi \cdot x(t)}) \cdot h(t) \).

Now, by using convolution theorem as given in (6) for fractional domain we have

\[ \{ F_{\alpha} x(t-\tau) \} (s) = e^{\frac{i}{2} \cot \phi \cdot \frac{x}{\alpha}} \cdot \{ F_{\alpha} e^{i \cot \phi \cdot x(t)} \} (s) \cdot \{ F_{\alpha} h(t) \} (s) \]

\[ = e^{\frac{i}{2} \cot \phi \cdot \frac{x}{\alpha}} e^{\frac{i}{2} \cot \phi \cdot \frac{(x^2 + \tau^2)}{\alpha}} \cdot \{ F_{\alpha} e^{i \cot \phi \cdot x(t)} \} (s) \cdot \{ F_{\alpha} h(t) \} (s) \]

\[ = e^{\frac{i}{2} \cot \phi \cdot \frac{x}{\alpha}} \cdot \{ F_{\alpha} e^{i \cot \phi \cdot x(t)} \} (s) \]

(7)

Note that this result (7) coincides with the shifting property for fractional Fourier transform in its new form as given in (4).
Thus output of the linear fractional shift invariant system is given by fractional convolution of the functions \( e^{it \cos \theta}x(t) \) and \( h(t) \), where \( h(t) \) is impulse response.

4. Conclusion

A new definition of linear fractional shift invariant system is introduced. This definition generalizes the concept of linear shift invariant system and Fourier transform convolution in the fractional domain.

References


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