Sharper Inequalities for Numerical Radius for Hilbert Space Operator

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Abstract: We give several sharp inequalities for the numerical radius of Hilbert space operators. It is shown that if A and B are bounded linear operators on complex Hilbert space \( H \), then

\[
\omega(A + B) \leq 2\alpha^2 \left( \left\| A^{\alpha\alpha} \right\| + \left\| B^{\alpha\alpha} \right\| + \left\| A^{\star(1-\alpha)\alpha} \right\| + \left\| B^{\star(1-\alpha)\alpha} \right\| \right)^{1/2} + \frac{1}{2} \left\| (A + B)^{\alpha\alpha} \right\|^{1/2},
\]

for \( 0 < r \leq 1 \) and \( \alpha \in (0,1) \), and if \( A \in M_n(\mathbb{C}) \), then

\[
w^2(A) \leq \frac{1}{4} \left( \left\| A + A^\star \right\|^2 + \left\| A - A^\star \right\|^2 \right),
\]

where \( \omega(\cdot) \) and \( \left\| \cdot \right\| \) are the numerical radius and the usual operator norm, respectively. In addition, from the first inequality, we obtain sharp inequalities.

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1. Introduction:

Let \( B(H) \) denotes the C*-algebra of all bounded linear operators on a complex Hilbert space \( H \) with inner product \( \langle \cdot, \cdot \rangle \). For \( A \in B(H) \), let \( \omega(A) \) and \( \| A \| \) denotes the numerical radius and the usual operator norm of \( A \), respectively. It is well known that \( \omega(\cdot) \) defines a norm on \( B(H) \) and that for every \( A \in B(H) \),
It has been shown in [3], that if \( A \in B(H) \), then
\[
w(A) \leq \frac{1}{2} \left( \| A \| + \| A^* \| \right)^{1/2}.
\] (2)

Also, it has been shown in [9], that if \( A, B \in B(H) \), then
\[
\| A + B \| \leq 2^{-r/2} \left( \| A \|^{2r} + \| B \|^{2r} + \| A^* \|^{2(1-\alpha)r} + \| B^* \|^{2(1-\alpha)r} \right)^{1/2},
\] for \( 0 < r \leq 1 \) and \( \alpha \in (0,1) \). (3)

In this paper, we present several sharp inequalities for the numerical radius of Hilbert space operators. In addition, we establish an inequality that refines the inequality (1) and the inequality (2). Moreover, by using the Clarkson's inequality we obtain if \( A \in M_n(\mathbb{C}) \), then
\[
w^2(A) \leq \frac{1}{4} \left( \| A + A^* \| + \| A - A^* \| \right).
\]

Fujii and Kubo [10], Kittaneh have been invoked numerical radius estimation see ([3] and [4]), Linden [6] and Yuri [1].

**Definition 1.1 [12]:** For a matrix, \( A \in B(H) \), then the numerical radius is \( \omega(A) \) of \( A \) defined as \( \omega(A) := \max \{ \| Ax \| : x \in H, x^* x = 1 \} \).

To achieve our goal we need the following well known lemmas.

**Lemma 1.2 [2]:** Let \( A \in B(H) \), and \( x \in H \) be unit any vector. Then
\[
\langle Ax, x \rangle \geq \langle A^* x, x \rangle \quad \text{for all } r \geq 1.
\]
\[
\langle Ax, x \rangle \geq \langle A^* x, x \rangle \quad \text{for all } 0 < r \leq 1, \text{ and } \| \langle Ax, x \rangle \| \leq \| A^* x, x \| \quad \text{for all } x \in H.
\]

**Lemma 1.3 [2]:** If \( a, b \in \mathbb{C} \), then
\[
2 \left( |a|^r + |b|^r \right) \leq |a + b|^r + |a - b|^r, \quad \text{for } r \geq 2 \text{ (the Clarkson's inequality)}.
\]

**2. Main Results:**

In this section, we present numerical radius inequality for the sum of two operators. In addition, from this inequality, we obtain several sharp inequalities for the numerical radius of Hilbert space operators.
**Theorem 2.1:** Let $A, B \in B(H)$, $0 < \alpha \leq 1$ and $r \geq 1$. Then

$$w(A + B) \leq 2^{-r \frac{2}{2}} \left( \|A\|^{2\alpha r} + \|B\|^{2\alpha r} + \|A^r + B^r\|^{2(1-\alpha)r} + \|A^r - B^r\|^{2(1-\alpha)r} \right)^{\frac{1}{r}} + \frac{1}{2} \|(A + B)^2\|^{\frac{1}{2}}. \quad (4)$$

**Proof:**
From the inequality (2), we have if $A, B \in B(H)$, then

$$w(A + B) \leq \frac{1}{2} \left( \|A + B\| + \|(A + B)^2\|^{\frac{1}{2}} \right). \quad (5)$$

By substituting the inequality (3) in (5) we obtain the result.

**Corollary 2.1:** Letting $\alpha = \frac{1}{2}$, and $r = 1$ in (4), we obtain

$$w(A + B) \leq \frac{1}{4} \left( \|A\| + \|B\| + \|A^r + B^r\|^{2(1-\alpha)r} \right) + \frac{1}{2} \|(A + B)^2\|^{\frac{1}{2}}. \quad (6)$$

**Corollary 2.2:** Letting $B = A$ in (4), we obtain

$$w(A) \leq 2^{-r \frac{2}{2}} \left( \|A\|^{2\alpha r} + \|A^r\|^{2(1-\alpha)r} \right)^{\frac{1}{r}} + \frac{1}{2} \|A^2\|^{\frac{1}{2}}. \quad (7)$$

**Corollary 2.3:** Letting $\alpha = \frac{1}{2}$, and $r = 1$ in (7), we obtain the inequality (2) (e.g. see [3]).

**Corollary 2.4:** Letting $\alpha = \frac{1}{2}$, and $r = 2$ in (7), we obtain

$$w(A) \leq \frac{1}{\sqrt{8}} \left( \|A\| + \|A^2\| \right)^{\frac{1}{2}} + \frac{1}{2} \|A^2\|^{\frac{1}{2}}. \quad (8)$$

The inequality (8) is sharper than the inequality (2) because,

$$\frac{1}{\sqrt{8}} \left( \|A\| + \|A^2\| \right)^{\frac{1}{2}} + \frac{1}{2} \|A^2\|^{\frac{1}{2}} \leq 2^{-3} \left( 2^{\frac{1}{2}} \|A\| \right) + \frac{1}{2} \|A^2\|^{\frac{1}{2}} = \frac{1}{2} \left( \|A\| + \|A^2\| \right).$$

In the following theorem, we present new generalized an upper bound for the numerical radius of all $n \times n$ complex matrices.

**Theorem 2.2:** If $A \in M_n(C)$, then

$$w'(A) \leq \frac{1}{4} \left( \|A + A^r\| + \|A - A^r\| \right). \quad (9)$$
Proof:

\[ |\langle Ax, x \rangle| \leq \left( \frac{1}{2} \left| \langle A^* x, x \rangle \right|^2 + \frac{1}{2} \left( \langle A^* x, x \rangle \right)^2 \right)^{\frac{1}{2}} \quad \text{(By C-S inequality)} \]

\[ |\langle Ax, x \rangle| \leq \left( \frac{1}{2} \left| \langle A x, x \rangle \right| + \frac{1}{2} \left( \langle A^* x, x \rangle \right)^2 \right)^{\frac{1}{2}}, \quad \text{for } r \geq 0. \]

\[ |\langle Ax, x \rangle| \leq \left( \frac{1}{2} \left( |\langle A x, x \rangle| + |\langle A^* x, x \rangle| \right)^2 \right)^{\frac{1}{2}}, \quad \text{for } r \geq 1 \text{(by using lemma 1.2)} \]

\[ 4|\langle Ax, x \rangle|^r \leq 4\left( \left( |\langle A^* x, x \rangle| + |\langle A^* x, x \rangle| \right) |x, x| \right)^{x}, \quad \text{for } r \geq 1 \text{(by using lemma 1.3)} \]

\[ 4\left| \langle Ax, x \rangle \right|^r \leq \left( |A + A^*|^r + |A - A^*|^r \right) |x, x|, \quad \text{for } r \geq 2. \]

For, \[ 4\left| \langle Ax, x \rangle \right|^r \leq \left( |A + A^*|^r + |A - A^*|^r \right) |x, x|, \quad \text{for } r \geq 2. \]

Now, by taking the maximum value over all unit vectors in \( \mathbb{C}^n \), so we get the result.

**Corollary 2.5:** Letting \( r = 2 \), in the previous theorem, gives

\[ w^2(A) \leq \frac{1}{4} \left( \left\| A + A^* \right\|^r + \left\| A - A^* \right\|^r \right). \quad (10) \]

**References**


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