Semigroup of Operators Commuting with Translations on Compact Commutative Hypergroups

Norbert Youmbi

Department of Mathematics, Saint Francis University
117 Evergreen Dr, Sullivan 114, Loretto, PA 15931, USA
nyoumbi@francis.edu

Abstract

One of the central results in the theory of semigroups of operators is that the semigroup $S = \{ T(\xi) : \xi > 0 \}$ and its infinitesimal generator $A$ are connected by an exponential formula in the form $T(\xi) = e^{\xi A}$. Let $H$ be a compact commutative hypergroup with dual space $\hat{H}$.

Let $U = C(H)$ or $L_p(H)$, it is shown that corresponding to each semigroup $S = \{ T(\xi) : \xi > 0 \}$ of operators on $U$, which commutes with translations, there is a semigroup $M = \{ E_\xi : \xi > 0 \}$ of $U$ -multipliers. Conversely a semigroup $M = \{ E_\xi : \xi > 0 \}$ of $U$ -multipliers determines a semigroup $S = \{ T(\xi) : \xi > 0 \}$ of operators on $U$, which commutes with translations.

Mathematics Subject Classification: 43A62

Keywords: Hypergroups, semigroup of operators, semigroup of multipliers

1 Introduction

A hypergroup is roughly speaking a locally compact Hausdorff space which has enough structure so that a convolution on the corresponding vector space of Radon measures makes it a Banach algebra. Hypergroups generalize in many ways locally compact groups.

Let $H$ be a compact commutative hypergroup with dual space $\hat{H}$. We denote by $C(H)$ the Banach space of all continuous complex valued functions on $H$ with uniform norm. Let $L_p(H), 1 \leq p < \infty$, has its usual meaning.

Let $U = C(H)$ or $L_p(H)$. Given $A \subset U$, we define by $\hat{A}$, the set of all Fourier transforms $\hat{f}$ of $f \in A$. A complex valued mapping $\varphi$ on the dual space $\hat{H}$ of $H$ is called an $(A, B)$-multiplier if and only if $\varphi \hat{f} \in B$ for each
A family $S = \{T(\xi) : \xi > 0\}$ of bounded linear operators on $U$ is said to be a semigroup of operators on $U$ if and only if

$$T(\xi_1 + \xi_2) = T(\xi_1)T(\xi_2) \text{ for all } \xi_1, \xi_2 > 0.$$  

The infinitesimal operator $A_0$ of $S$ is defined as the limit in norm as $\eta \to 0^+$ of $A_\eta f = \frac{1}{\eta}[T(\eta) - I]f$ whenever it exists.

In general $A_0$ is an unbounded linear operator; however the domain of $A_0$ is dense in the union of the range spaces of $\{T(\xi) : \xi > 0\}$. The operator $A_0$ is in general not closed; its closure $\hat{A}_0$, when it exists, will be called the infinitesimal generator of $S$.

A comprehensive account of semigroups of operators on Banach spaces can be found in Hille and Phillips [5], where all undefined terms used in this work in connection with such semigroups are explained.

The results of this paper can be summarized as follows. Given a semigroup $S = \{T(\xi) : \xi > 0\}$ of operators on $U$, which commutes with translations, we associate a semigroup $\mathcal{M} = \{E_\xi : \xi > 0\}$ of $U$-multipliers. We show that if $T(\xi)$ is weakly measurable, then there exists a subset $\hat{H}_0$ of $\hat{H}$ and a complex valued mapping $\varphi : \chi \mapsto \varphi(\chi)$ from $\hat{H}_0$ to $\mathbb{C}$ such that $E_\xi(\chi) = e^{\varphi(\chi)\xi}$ if $\chi \in \hat{H}_0$ and $E_\xi(\chi) = 0$ if $\chi \notin \hat{H}_0$ for each $\xi > 0$. Conversely a semigroup $\mathcal{M} = \{E_\xi : \xi > 0\}$ of $U$-multipliers determines a semigroup $S = \{T(\xi) : \xi > 0\}$ of operators on $U$, which commutes with translations. Moreover if for a fixed subset $\hat{H}_0$ of $\hat{H}$ and a complex valued mapping $\varphi : \chi \mapsto \varphi(\chi)$ on $\hat{H}_0$, we define $E_\xi(\chi) = e^{\varphi(\chi)\xi}$ if $\chi \in \hat{H}_0$ and $E_\xi(\chi) = 0$ if $\chi \notin \hat{H}_0$ such that $\mathcal{M} = \{E_\xi : \xi > 0\}$ is a semigroup of $U$-multipliers, then the corresponding semigroup of operators $S$ is strongly continuous. Finally we prove that if $A_0$ ($A$) is the infinitesimal operator (generator) of $S$, then $\varphi$ is a $(D(A_0), U)$-multiplier ($(D(A), U)$-multiplier). These results generalize those of Hille and Phillips [5] for the circle group and Babalola and Olubummo [1] for compact abelian groups.

2 Preliminaries

Let $M(H)$ be the set of finite regular Borel measures on $H$; $M_1(H)$ be the space of probability measures. If $\mu \in M(H)$ then $\text{Supp}(\mu) = \{x \in H : \text{if } V \text{ is any open set containing } x \text{ then } \mu(V) > 0\}$. If $x \in H$, $\delta_x$ is the point mass at $x$. An unspecified topology on $M_+(H)$ is the cone topology. In this paper we are using the following definition of a hypergroup.
2.1 Definition

A nonempty locally compact Hausdorff space $H$ will be called a hypergroup if the following conditions are satisfied:

$(H_1)$ $(M(H),+,\ast)$ is a Banach algebra.

$(H_2)$ For all $x,y \in H$, $\delta_x \ast \delta_y$ is a probability measure with compact support.

$(H_3)$ The mapping $(x,y) \mapsto \delta_x \ast \delta_y$ of $H \times H$ into $M_1(H)$ is continuous.

$(H_4)$ The mapping $(x,y) \mapsto \text{Supp}(\delta_x \ast \delta_y)$ of $H \times H$ into $\mathcal{C}(H)$ is continuous where $\mathcal{C}(H)$ is the space of compact subsets of $H$ endowed with the Michael topology, that is the topology generated by the subbasis of all $\mathcal{C}_U(V) = \{ C \in \mathcal{C}(S) : C \cap U \neq \emptyset \text{ and } C \subset V \}$ where $U$ and $V$ are open subsets of $S$.

$H_5$ There exists $e \in H$ such that $\delta_x \ast \delta_e = \delta_e \ast \delta_x = \delta_x \ \forall x \in H$.

$H_6$ There exists a topological involution $-$ (a homeomorphism) from $H$ onto $H$ such that $(x^-)^- = x \ \forall x \in H$, with $(\delta_x \ast \delta_y)^- = \delta_y^- \ast \delta_x^-$ and $e \in \text{Supp}(\delta_x \ast \delta_y)$ if and only if $x = y^-$ where for any Borel set $B$, $\mu^-(B) = \mu(\{ x^- : x \in B \})$.

Remarks

(i) If $\delta_x \ast \delta_y = \delta_y \ast \delta_x$ for all $x,y \in H$ we say that $(H,\ast)$ is a commutative hypergroup.

(ii) The convolution $\ast$ on $M(H)$ is defined by

$$\mu \ast \nu(f) = \int_H f \, d\mu \ast \nu = \int_H \mu(dx) \int_H \nu(dy) \int_H f \, d\delta_x \ast \delta_y$$

for all $f \in C_b(H)$.

2.2 Examples

1. If $(G,\cdot)$ is a locally compact Hausdorff group, then with convolution defined by $\delta_x \ast \delta_y = \delta_{xy}$, $(G,\ast)$ is a hypergroup. Also if a hypergroup is such that the convolution of two point masses is a point mass then it is a topological group.
2. Consider the segment $[0, 1]$ with convolution defined by

$$
\delta_r \ast \delta_s = \frac{1}{2} \delta_{|r-s|} + \frac{1}{2} \delta_{1-|r-s|}
$$

for all $r, s \in [0, 1]$, then $([0, 1], \ast)$ is a compact commutative hypergroup.

For a detailed discussion on hypergroups see Bloom and Heyer [2], Jewett [6], Dunkl [3], Spector [9].

2.3 Definition

Let $(H, \ast)$ be a hypergroup. If $f$ is a Borel function on $H$ and $x, y \in H$ then we define

$$
f(x \ast y) = f_x(y) = f^y(x) = \int_H f(\delta_x \ast \delta_y)
$$

if this integral exists, even when it is not finite. $f_x$ is called the left translation of $f$ by $x$, and $f^y$ is called the right translation of $f$ by $y$.

2.4 Definition

A character $\chi$ on a hypergroup $H$ is a continuous complex-valued mapping on $H$ which is not identically zero and satisfies

$$
\int_H \chi d\delta_x \ast \delta_y = \chi(x)\chi(y)
$$

for all $x, y \in H$. A character $\chi$ is said to be Hermitian if and only if $\chi(x^-) = \chi(x)$. The set of all Hermitian characters on $H$ is denoted by $\hat{H}$.

2.5 Example

Let $H = [0, +\infty)$ and $\varphi_\lambda(x) = \cos \lambda x \ \lambda \in [0, +\infty)$ then we have the relation

$$
\varphi_\lambda(x)\varphi_\lambda(y) = \frac{1}{2} [\varphi_\lambda(x + y) + \varphi_\lambda(x - y)]
$$

for all $\lambda \in [0, +\infty)$ since $\varphi_\lambda$ is an even function, this relation is equivalent to

$$
\varphi_\lambda(x)\varphi_\lambda(y) = \frac{1}{2} [\varphi_\lambda(x + y) + \varphi_\lambda(|x - y|)] = \frac{1}{2} [\delta_{x+y}(\varphi_\lambda) + \delta_{|x-y|}(\varphi_\lambda)]
$$

Let $\sigma_{x,y} = \frac{1}{2} [\delta_{x+y} + \delta_{|x-y|}]$ then $\{\varphi_\lambda\}$ satisfies the product formula

$$
\varphi_\lambda(x)\varphi_\lambda(y) = \int \varphi_\lambda(z)\sigma_{x,y}(dz).
$$
Now given two Radon measures $\mu$ and $\nu$ on $H$ we can define a convolution

$$\mu \ast \nu(f) = \int \int f(x,y) \mu(dx) \nu(dy)$$

for all $f \in C_c(H)$. With this convolution, $M(H)$ is a Banach algebra [10].

If we define a convolution of point masses by

$$\delta_x \ast \delta_y = \frac{1}{2} [\delta_{x+y} + \delta_{|x-y|}]$$

$(H, \ast)$ is a hypergroup with identity element 0, the involution is the identity function. Furthermore the characters of $(H, \ast)$ are the orthogonal functions $\{\varphi_{\lambda}\}$, $\lambda \in [0, +\infty)$.

2.6 Definition

Let $H$ be a locally compact hypergroup. A measure $m$ not necessarily finite, will be called left subinvariant (invariant) if $\delta_x \ast m$ is defined and $\delta_x \ast m \leq m$ ($\delta_x \ast m = m$) for all $x \in H$. (Right invariant measures are defined in the same way).

2.7 Example

The invariant measure on the hypergroup $(H, \ast)$ of example 2.5 is the Lebesgue measure. In general if a system of orthogonal functions with respect to a measure $m$, has a product formula which defines a hypergroup $H$ then the measure $m$ is the invariant measure of the hypergroup $H$.

2.8 Remarks


2. Unlike in the group case, $\hat{H}$ is not always a hypergroup even in the commutative case see Jewett [6]. From now on, we will assume that $\hat{H}$ is a hypergroup with invariant measure $\pi$ such that $\text{supp}(\pi) = \hat{H}$.

2.9 Definition

Let $(H, \ast)$ be a commutative hypergroup with invariant measure $m$. The Fourier Stieltjes transform $\hat{\mu}$ of $\mu$ is defined on $\hat{H}$ by

$$\hat{\mu}(\chi) = \int_{\hat{H}} \chi d\mu$$
and for all $f \in L_1(H)$, the Fourier transform $\hat{f}$ of $f$ with respect to $m$ is defined by
\[
\hat{f}(\chi) = \int_{H} f \overline{\chi} dm
\].

### 2.10 Lemma

Let $H$ be a compact commutative hypergroup and $E$ a $U$-multiplier. Define the operator $T : U \to U$ by $\widehat{T(f)} = E\hat{f}$ for $f \in U$. Then $T$ is a bounded linear operator on $U$.

**Proof**

Since $E$ is a $U$-multiplier, $E\hat{f} \in U$ for all $f \in U$ so by the uniqueness and the linearity property of the Fourier transform, $T$ is a well-defined linear operator.

Now let $f_n$ be a sequence of elements of $U$ converging in $U$-norm to $f \in U$. Suppose also that $T(f_n)$ converges in $U$-norm to $g \in U$. Then $\widehat{T(f_n)}$ converges to $\hat{g}$ in the $\hat{U}$-norm. By definition, $\widehat{T(f)} = E\hat{f}$ for $f \in U$. We also have
\[
\|E\hat{f}_n - E\hat{f}\|_{\hat{U}} \leq \|E(\hat{f}_n - \hat{f})\|_{\hat{U}} \leq \|E\|_{\hat{U}} \|\hat{f}_n - \hat{f}\|_{\hat{U}} \leq \|E\|_{\hat{U}} \|f_n - f\|_U.
\]
because the Fourier transform is norm decreasing (Jewett [6],7.3). It follows that $\widehat{T(f_n)}$ converges to $\widehat{T(f)}$, so $T(f) = g$, hence $T$ has a closed graph and by the closed graph theorem, $T$ is bounded.

**Remark**

The next theorem characterizes multipliers on $L_1(H)$, where $H$ is a locally compact commutative hypergroup.

### 2.11 Wendel’s Theorem [7]

Let $H$ be a locally compact commutative hypergroup. Suppose $T : L_1(H) \to L_1(H)$ is a bounded linear Operator. Then the following statements are equivalent:

**i.** $T$ commutes with translations, that is $T(f^x) = T(f)^x$ for all $x \in H$

**ii.** $T(f \ast g) = T(f) \ast g$ for each $f, g \in L_1(H)$

**iii.** There exists a unique transformation $\varphi$ on $\hat{H}$ such that $\widehat{T(f)} = \varphi \hat{f}$ for each $f \in L_1(H)$.

**iv.** There exists a unique measure $\mu \in M(H)$ such that $\widehat{T(f)} = \mu \hat{f}$ for each $f \in L_1(H)$.
v. There exists a unique measure \( \mu \in M(H) \) such that \( T(f) = f * \mu \) for each \( f \in L_1(H) \).

3 Semigroup of operators and semigroup of multipliers on \( \mathcal{U} \)

We now prove our main results.

3.1 Theorem

Let \( \mathcal{S} = \{ T(\xi) : \xi > 0 \} \) be a semigroup of bounded linear operators on \( \mathcal{U} \). Suppose that for each \( \xi > 0 \), the operator \( T(\xi) \) commutes with translations. Then \( \mathcal{S} \) defines a semigroup \( \mathcal{M} = \{ E_\xi : \xi > 0 \} \) of \( \mathcal{U} \)-multipliers such that

i. For each \( \xi > 0 \), \( E_\xi \hat{f} = \widehat{T(\xi)\hat{f}} \) for each \( f \in \mathcal{U} \); and

ii. \( E_{\xi_1+\xi_2}(\chi) = E_{\xi_1}(\chi)E_{\xi_2}(\chi) \), \( \xi_1, \xi_2 > 0 \) and \( \chi \in \hat{H} \)

If moreover, \( T(\xi) \) is weakly measurable, then there exists a subset \( \hat{H}_0 \) of \( \hat{H} \) and a mapping \( \varphi : \chi \mapsto \varphi(\chi) \) of \( \hat{H}_0 \) into \( \mathbb{C} \) such that

\[
E_\xi(\chi) = \begin{cases} 
\varepsilon^{\varphi(\chi)\xi} & \text{if } \chi \in \hat{H}_0, \\
0 & \text{if } \chi \notin \hat{H}_0.
\end{cases}
\]

for each \( \xi > 0 \).

Proof:

i. For all \( \xi > 0 \), \( T(\xi) \) is a bounded linear operator on \( \mathcal{U} \) which commutes with translation and from Theorem 2.11, there is a unique \( E_\xi \) on \( \hat{H} \) such that \( \widehat{T(\xi)f} = E_\xi \hat{f} \) for all \( f \in \mathcal{U} \).

ii. For all \( \xi_1, \xi_2 > 0 \), \( [T(\xi_1 + \xi_2)\hat{f}] = E_{\xi_1+\xi_2}\hat{f} \) But

\[
[T(\xi_1 + \xi_2)\hat{f}] = [T(\xi_1)T(\xi_2)\hat{f}] = E_{\xi_1}[T(\xi_2)\hat{f}] = E_{\xi_1}(E_{\xi_2}\hat{f})
\]

Since \( f \) is arbitrary, \( E_{\xi_1+\xi_2} = E_{\xi_1}E_{\xi_2} \) that is, \( E_{\xi_1}(\chi)E_{\xi_2}(\chi) = E_{\xi_1+\xi_2}(\chi) \) which proves (ii).

Suppose now that \( T(\xi) \) is weakly measurable, then for each continuous linear functional \( \psi \) on \( \mathcal{U} \) and for each \( f \in \mathcal{U} \), the mapping \( \xi \mapsto \psi(T(\xi)f) \) from \( \mathbb{R} \to \mathbb{C} \) is Lebesgue measurable. In particular if for each \( \chi \in \hat{H} \) we define \( \psi_\chi \) by \( \psi_\chi(f) = \hat{f}(\chi) \) , \( f \in \mathcal{U} \), then \( \psi_\chi \) is a continuous linear functional on \( \mathcal{U} \) such that the mapping
\( \xi \mapsto \psi_\chi(T(\xi)\chi) = E_\xi(\chi)\hat{\chi}(\chi) \) is measurable. It follows that for each \( \chi \), \( E_\xi(\chi) \) is measurable. Since
\[
E_{\xi_1 + \xi_2}(\chi) = E_{\xi_1}(\chi)E_{\xi_2}(\chi),
\]
\( E_\xi(\chi) \) is a measurable character and from Hille and Phillips [5] corollary to theorem 4.17.3, it follows that either \( E_\xi(\chi) = 0 \) or \( E_\xi(\chi) = e^{\varphi(\chi)\xi} \), for some complex numbers \( \varphi(\chi) \). Set \( \hat{H}_0 = \{ \chi \in \hat{H} : E_\xi(\chi) \neq 0 \} \) to complete the proof.

Now Suppose that \( \hat{H}_0 \) is a fixed subset of \( \hat{H} \) and let \( \varphi : \chi \mapsto \varphi(\chi) \) be a mapping from \( \hat{H}_0 \) into \( \mathbb{C} \). Put
\[
E_\xi(\chi) = \begin{cases} 
e{e^{\varphi(\chi)\xi}} & \text{if } \chi \in \hat{H}_0, \\ 0 & \text{if } \chi \notin \hat{H}_0. \end{cases}
\]
Assume that \( E_\xi \) as defined here is a \( \mathcal{U} \)-multiplier, then we have the following theorem.

### 3.2 Theorem

For each \( \xi > 0 \), define a mapping \( T(\xi) \) of \( \mathcal{U} \) into itself by \( \hat{T(\xi)f} = E_\xi\hat{f}, f \in \mathcal{U} \) then

i. \( S = \{ T(\xi) : \xi > 0 \} \) defines a semigroup of bounded linear operators on \( \mathcal{U} \), the elements of which commute with translations and are continuous in the strong operator topology for \( \xi > 0 \).

ii For each \( f \in D(A_0) \) and \( \chi \notin \hat{H}_0 \) we have \( \hat{f}(\chi) = 0 \) where \( A_0 \) denotes the infinitesimal operator of \( S \) and \( D(A_0) \) is the domain of \( A_0 \). Moreover, \( \varphi \) is a \( (D(A_0), \mathcal{U}) \)-multiplier since \( \hat{A}_0\hat{f} = \varphi\hat{f} \) for all \( f \in D(A_0) \).

iii If \( S \) is of class \( (A) \), then \( \hat{H}_0 = \hat{H} \) and
\[
D(A) = \{ f \in \mathcal{U} : \varphi\hat{f} \in \hat{\mathcal{U}} \}
\]
that is \( \varphi \) is a \( (D(A), \mathcal{U}) \)-multiplier and moreover \( \hat{A}f = \varphi\hat{f} \) for all \( f \in D(A) \), where \( A \) is the infinitesimal generator of \( S \).

### Proof:

i. From Lemma 2.10, \( T(\xi) \) is a bounded linear operator for each \( \xi > 0 \), moreover we have
\[
[T(\xi_1 + \xi_2)\hat{f}] = E_{\xi_1 + \xi_2}\hat{f} = E_{\xi_1}(E_{\xi_2}\hat{f}) = E_{\xi_1}([T(\xi_2)\hat{f}]) = [T(\xi_1)T(\xi_2)\hat{f}]
\]
Suppose that $\text{Semigroup of operators commuting with translations}$. 

Now let's prove that $T(\xi)$ is continuous in the strong operator topology for $\xi > 0$. First suppose that $t \in I(H)$, the set of all finite complex linear combination of continuous characters on $H$. Thus $t$ is of the form $t = \sum_{i=1}^{n} \alpha_i \chi_i$, the orthogonality of $I(H)$ implies $T(\xi)$ is defined by

$$T(\xi)t(x) = \sum_{i=1}^{n} \alpha_i e^{\phi(x)\xi}\chi_i(x), \quad x \in H.$$ 

We then have,

$$\|T(\xi)t - T(\xi_0)t\| = \| \sum_{i=1}^{n} [\alpha_i e^{\phi(x)\xi} - \alpha_i e^{\phi(x)\xi_0}] \chi_i \| \leq \sum_{i=1}^{n} |\alpha_i| |e^{\phi(x)\xi} - e^{\phi(x)\xi_0}| \to 0 \text{ as } \xi \to \xi_0.$$

Suppose now that $f$ is arbitrarily chosen in $\mathcal{U}$ and let $\epsilon > 0$ be given. Then, there exists $t \in I(H)$ such that $\|f - t\| < \epsilon$. Since

$$\|T(\xi)f - T(\xi)t\| \leq \|T(\xi)\|\|f - t\|$$

for each $\xi > 0$, $T(\xi)$ is strongly measurable by [5] theorem 3.5.4. Hence by [5] theorem 10.2.3, $T(\xi)$ is continuous in the strong operator topology for $\xi > 0$. This completes the proof of (i).

**ii.** Let $f \in D(A_0)$, then in the norm topology,

$$\lim_{\eta \to 0^+} \frac{1}{\eta}(T(\eta)f - f)$$

exists. Let this limit be $g$, then $g = A_0f$. For each $\chi$

$$\frac{1}{\eta} \hat{T(\eta)f}(\chi) - \hat{f}(\chi) \to g(\chi)$$

and since $\hat{T(\eta)f} = E_{\eta}\hat{f}$ we have $\frac{1}{\eta}[E_{\eta}(\chi) - 1] \hat{f}(\chi) \to g(\chi)$ as $\eta \to 0^+$.

As a consequence of the definition of $E_{\eta}$, it follows that, $\hat{f}(\chi) = 0$ if $\chi \notin \hat{H}_0$ and $A_0\hat{f}(\chi) = \varphi\hat{f}(\chi)$ for all $\chi \in \hat{H}_0$. Which proves (ii).

**iii.** Suppose that $\{T(\xi) : \xi > 0\}$ is of class $(A)$ with infinitesimal generator $A = \hat{A}_0$, the smallest closed extension of its infinitesimal operator $A_0$. Then $\mathcal{U}_0 = \{T(\xi)f : f \in \mathcal{U}, \xi > 0\}$ and $D(A_0)$ are dense in $\mathcal{U}$. Suppose there exists $\chi_0 \in \hat{H}$ such that $\chi_0 \notin \hat{H}_0$; choose $f \in \mathcal{U}$ such that $\hat{f}(\chi_0) \neq 0$. Given $\epsilon > 0$, there exists an $f' \in D(A_0)$ such that

$$\|f' - f\| < \epsilon \text{ then } |\hat{f}'(\chi_0) - \hat{f}(\chi_0)| \leq \|f' - f\| < \epsilon$$

and since this is true for all $\epsilon > 0$, $\hat{f}'(\chi_0) = \hat{f}(\chi_0) = 0$ a contradiction. Hence $\hat{H}_0 = \hat{H}$. Finally let

$$\omega_0 = \inf \|T(\xi)\| = \lim_{\xi \to \infty} \frac{1}{\xi} \log \|T(\xi)\|,$$
that is $\mathcal{S}$ is of type $\omega_0$. For $\lambda$ with $Re(\lambda) > \omega_0$, let $R(\lambda; A)$ denote the resolvent of the infinitesimal generator $A$ of $\mathcal{S}$ then there exists a $\omega_1 > \omega_0$ such that

$$R(\lambda; A)f = \int_0^\infty e^{-\lambda \xi} T(\xi) f d\xi, \quad f \in \mathcal{U}_0, \quad Re(\lambda) > \omega_1.$$  

Since, $\forall \chi \in \hat{H}$, the mapping $f \mapsto \hat{f}(\chi)$ is a bounded linear functional on $\mathcal{U}$, we have for all $f \in \mathcal{U}_0$,

$$(R(\lambda; A)f)(\chi) = \int_0^\infty e^{-\lambda \xi} \hat{T}(\xi) \hat{f}(\chi) d\xi =$$

$$\int_0^\infty e^{-\lambda \xi} \varphi(\xi) \hat{f}(\chi) d\xi = \frac{1}{\varphi(\chi) - \lambda} e^{\varphi(\xi) - \lambda \xi} \hat{f}(\chi)\big|_{\xi=0} = (\lambda - \varphi(\chi))^{-1} \hat{f}(\chi)$$

for each $\chi \in \hat{H}$. Since $\mathcal{U}_0$ is dense in $\mathcal{U}$, we have

$$[(R(\lambda; A)f)](\chi) = (\lambda - \varphi(\chi))^{-1} \hat{f}(\chi)$$

for all $f \in \mathcal{U}$ with $Re(\lambda) > \omega_1$.

Let $\lambda > \omega_1$ be fixed and suppose that $f \in D(A)$. Then there exists a $g \in \mathcal{U}$ such that $f = R(\lambda; A)g$, and we have, for each $\chi \in \hat{H}$,

$$\hat{A}f(\chi) = [\lambda R(\lambda; A)g - \hat{g}(\chi) = [\lambda R(\lambda; A)\hat{g}(\chi) - \hat{g}(\chi) =$$

$$\lambda(\lambda - \varphi(\chi))^{-1} \hat{g}(\chi) - \hat{g}(\chi) = (\frac{\lambda}{\lambda - \varphi(\chi)}) - 1) \hat{g}(\chi)$$

$$\frac{\lambda - \lambda + \varphi(\chi)}{\lambda - \varphi(\chi)} \hat{g}(\chi) = \varphi(\chi) \hat{R}(\lambda; A)g(\chi) = \varphi(\chi) \hat{f}(\chi)$$

Thus whenever $f \in D(A)$, $\varphi \hat{f} \in U$.

Conversely, suppose that $f$ is an element of $\mathcal{U}$ such that $\varphi \hat{f} \in \hat{U}$. This means that there exists an $h \in \mathcal{U}$ such that $\varphi(\chi) \hat{f}(\chi) = h(\chi)$ for all $\chi \in \hat{H}$. Then $g = \lambda f - h \in \mathcal{U}$ and, for all $\chi \in \hat{H}$,

$$[R(\lambda; A)g](\chi) = (\lambda - \varphi(\chi))^{-1} \hat{g}(\chi) = [\lambda - \varphi(\chi)]^{-1} [\lambda \hat{f}(\chi) - \varphi(\chi) \hat{f}(\chi)] =$$

$$[\lambda - \varphi(\chi)]^{-1} \hat{f}(\chi)$$

Which implies that $R(\lambda; A)g = f$ which also implies that $f \in D(A)$. This complete the proof of the theorem.

### 3.3 Remark

Examples are the Fourier Stieltjes transform of the convolution semigroups $\{\mu_t : t > 0\}$ on hypergroups, studied by Lasser [8].
3.4 Theorem (The case of $L_2(H)$)

Let $S = \{T(\xi) : \xi > 0\}$ be a semigroup of bounded linear operators on $L_2(H)$. Suppose that the operator $T(\xi)$ commutes with translations and is weakly measurable for $\xi > 0$. Then $S$ defines a semigroup $M = \{E_\xi : \xi > 0\}$ of $L_2(H)$-multipliers such that

i. $\hat{T(\xi)f} = E_\xi \hat{f}$ for all $f \in L_2(H)$, $\xi > 0$, and there exists a subset $\hat{H}_0$ of $\hat{H}$ and a mapping $\varphi : \chi \mapsto \varphi(\chi)$ of $\hat{H}_0$ into $\mathbb{C}$ such that

$$E_\xi(\chi) = \begin{cases} e^{\varphi(\chi)\xi} & \text{if } \chi \in \hat{H}_0, \\ 0 & \text{if } \chi \notin \hat{H}_0; \end{cases}$$

for each $\xi > 0$.

ii. $\|T(\xi)\|_2 = e^{\nu \xi}$, where $\nu = \sup_{\chi \in \hat{H}_0} \text{Re}(\varphi(\chi))$;

iii. If $T(0)$ is given by

$$\hat{T(0)f}(\chi) = \begin{cases} \hat{f}(\chi) & \text{if } \chi \in \hat{H}_0, \\ 0 & \text{if } \chi \notin \hat{H}_0, \end{cases}$$

then $T(\xi)$ is strongly continuous for $\xi \geq 0$ and if $\hat{H}_0 = \hat{H}$, then $S$ is of class $(C_0)$, that is

$$\lim_{\xi \to 0^+} T(\xi)f = f \text{ for all } f \in L_2(H)$$

Proof

i. follows as a particular case of theorem 3.1.

ii. By (i), it is obvious that $E_\xi$ is a bounded operator on $\hat{H}$ for each $\xi > 0$ and from $[6]$ 7.3I, $f \mapsto \hat{f}$ is a norm preserving isomorphism of $L_2(H)$ onto $l_2(\hat{H})$. It follows that

$$\|T(\xi)\|_2 = \sup_{\chi \in \hat{H}} |E_\xi(\chi)| = \sup_{\chi \in \hat{H}_0} |e^{\varphi(\chi)\xi}| = e^{\nu \xi}$$

where $\nu = \sup_{\chi \in \hat{H}_0} \text{Re}(\varphi(\chi))$.

iii From theorem 3.1, $T(\xi)$ is strongly continuous for $\xi > 0$. Now for each $f \in L_2(H)$,

$$\|[(T(\xi) - T(0))f]\|_2^2 = \|[T(\xi) - T(0)]\hat{f}\|_2^2 = \sum_{\chi \in \hat{H}_0} |e^{\varphi(\chi)\xi} - 1|^2 |\hat{f}(\chi)|^2$$

But $|e^{\varphi(\chi)\xi} - 1| \to 0$ as $\xi \to 0^+$ so $\|[(T(\xi) - T(0))f]\|_2^2 \to 0$ as $\xi \to 0^+$. Thus $T(\xi)$ is strongly continuous at $\xi = 0$. Now if $\hat{H}_0 = \hat{H}$, then
\[ \hat{T}(0)\hat{f}(\chi) = \hat{f}(\chi) \] for all \( f \in L_2(H) \) and \( \chi \in \hat{H} \). Therefore \( T(0) \) is the identity operator; and so

\[ \lim_{\xi \rightarrow 0^+} T(\xi)f = f \] for all \( f \in L_2(H) \), and the theorem is proved completely.

### 3.5 Theorem

If \( F = \{ \lambda \} \) is a closed subset of \( \mathbb{C} \) with \( \text{Re}(\lambda) \leq \alpha < \infty \), then there exists a semigroup \( S = \{ T(\xi) : \xi > 0 \} \) of class \( (C_0) \) on \( L_2(H) \) whose elements commutes with translations and such that the spectrum \( \sigma(A) = F \). Where \( A \) is the infinitesimal generator of \( S \).

**Proof**

Set \( \hat{H}_0 = \hat{H} \). Choose a complex-valued mapping \( \varphi \) on \( \hat{H} \) such that \( F \) is the closure of \( \{ \varphi(\chi) : \chi \in \hat{H} \} \). Define \( T(\xi), \xi > 0 \), by theorem 3.4(i). Then by theorem 3.4(iii), \( S \) is of class \( (C_0) \). And by equation (1) of theorem 3.2, the resolvent \( R(\lambda; A) \) of its infinitesimal generator \( A \) is given by

\[ [(R(\lambda; A)f)(\chi)](\chi) = (\lambda - \varphi(\chi))^{-1}\hat{f}(\chi), \chi \in \hat{H} \]

Consequently, \( \lambda \notin \sigma(A) \) if and only if \( \inf_{\chi \in \hat{H}}|\lambda - \varphi(\chi)| > 0 \). Hence \( \sigma(A) = F \), which proves the theorem.

### References


Received: October, 2008