Abstract

Replacing the definition, a new method is presented in order to show whether a function is belongs to the class \((p,q)\)-Ho(I) of functions \(f: I \to R\).

1. Introduction

A mapping \(f: I \subset [0,\infty) \to [0,\infty)\), \((where \ I = (0,\infty), [0,\infty), (0,1), (0,1), (0,1), \ or \ [0,1])\) is called \((p,q)\)-Holder type on I, where \(p,q\) are fixed with the property that \(1/p + 1/q = 1\), if

\[
\frac{f(xy)}{f(x)^{1/p}f(y)^{1/q}} \leq \text{for all } x, y \in I.
\]

This class of functions is denoted by \((p,q)\)-Ho(I), see [1].

In [1], many examples have been given belonging to the class \((p,q)\)-Ho(I), with their proofs. In this note we give a different easy method in order to check whether a function belongs to this class or not. In what follows we give the following

2. Main Result

**Theorem 2.1.** Let \(f: I \subset [0,\infty) \to [0,\infty)\), \(p,q > 1, 1/p + 1/q = 1\). If

\[
f''(x) + \frac{f'(x)}{x} - \left(\frac{f'(x)}{f(x)}\right)^2 \geq 0,
\]

then \(f \in (p,q)\)-Ho(I).

**Proof.** We have to show that
\[ f(xy) \leq \frac{1}{p} \left( x^p \right) \frac{1}{q} \left( y^q \right), \text{ or equivalently } f \left( \frac{1}{x^p} \frac{1}{y^q} \right) \leq \frac{1}{p} \left( x^p \right) \frac{1}{q} \left( y^q \right). \]

Set \( F(x) = \frac{1}{p} \left( x^p (x) \frac{1}{q} \left( f(y) \right) - f(x^p y^q) \right). \) On keeping \( y \) fixed and leaving \( x \) variate, we have

\[
F'(x) = \frac{1}{p} \left[ \left( \frac{f(y)}{f(x)} \right) \frac{1}{q} \left( f'(x) - \left( \frac{y}{x} \right) \frac{1}{q} \left( x^p y^q \right) \right) \right] = 0, \text{ if } x = y.
\]

\[
F^*(x) = \frac{1}{p} \left[ \left( \frac{f(y)}{f(x)} \right) \frac{1}{q} \left( f'(x) \right)^2 + \frac{1}{q} \left( \frac{y}{x} \right)^{\frac{1}{q}} \frac{1}{x} f^* \left( \frac{1}{x^p} \frac{1}{y^q} \right) \right]
\]

and

\[
[F^*(x)]_{x=y} = \frac{1}{pq} \left( f^*(x) + \frac{f'(x)}{x} \right) = \frac{1}{pq} \left( f^*(x) \right)^2.
\]

This shows that \( F \) attains its minimum when \( x = y \), which is zero. Therefore \( F(x) \geq F(0) = 0 \).

### 3. Applications

**Corollary 3.1.** Let \( g : R \rightarrow R \) be a convex mapping and put \( f : (0, \infty) \rightarrow (0, \infty) \) be given by \( f(x) = e^{g(\ln x)} \). Then \( f \in (p,q) - Ho(0,\infty), \) for all \( p,q \) as above.

**Proof.** It is not difficult to check that

\[
f^*(x) + \frac{f'(x)}{x} - \left( \frac{f'(x)}{x} \right)^2 = \frac{1}{x^2} g^*(\ln x) e^{x \ln x}.
\]

Clearly the above is nonnegative as \( g^* > 0 \), being convex.

**Corollary 3.2.** Let \( f,g : (0,\infty) \rightarrow (0,\infty) \) be defined by \( f(x) = x, \) \( g(x) = e^x \). Then \( f,g \in (p,q) - Ho(0,\infty) \) for all \( p,q \) as above.

**Proof.** We have

\[
f^*(x) + \frac{f'(x)}{x} - \left( \frac{f'(x)}{x} \right)^2 = 0,
\]

\[
g^*(x) + \frac{g'(x)}{x} - \left( \frac{g'(x)}{x} \right)^2 > 0.
\]
Corollary 3.3. Suppose \( f : (0, \infty) \to (0, \infty) \) is logarithmically convex and monotonic non-decreasing on \((0, \infty)\). Then \( f \in (p,q) - Ho(0,\infty) \) for all \( p, q \) as above.

Proof. By the hypothesis, we have \( f, f', (\ln f)'' \geq 0 \). Therefore

\[
f''(x) + \frac{f'(x)}{x} - \frac{(f'(x))^2}{f(x)} = f(x)(\ln f(x))'' + \frac{f'(x)}{x} \geq 0.
\]

Let \( f \) be absolutely monotonic mapping on \((0, \infty)\), i.e., \( f^{(k)}(t) \geq 0, t \in (0,\infty), k \in \mathbb{N} \), and \( f \) is continuous on \([0,\infty)\). It is known that (see [2, p.160]) if \( f \) is as above, then \( f \) can be represented by

\[
f(x) = \int_0^\infty e^{\sigma(t)} d\sigma(t), \quad x \in (0,\infty),
\]

where \( \sigma \) is a bounded and non-decreasing mapping on \([0,\infty)\) and the integral converges for \( 0 \leq x < \infty \). Moreover, we have

\[
f^{(j)}(x) = \int_0^\infty t^j e^{\sigma(t)} d\sigma(t) \quad \text{for all } x \in [0,\infty), \; j \in \mathbb{N}.
\]

Corollary 3.4. Let \( f \) be absolutely monotonic on \((0, \infty)\). Then for all \( j \in \mathbb{N} \), \( f^{(j)} \in (p,q) - Ho(0,\infty) \) for all \( p, q \) as above.

Proof. Set

\[
G(x) = f^{(j)}(x) = \int_0^\infty t^j e^{\sigma(t)} d\sigma(t).
\]

Then, we have

\[
G''(x) + \frac{G'(x)}{x} - \frac{(G'(x))^2}{G(x)} = \int_0^\infty t^{j+2} e^{\sigma(t)} d\sigma(t) + \frac{1}{x^j} \int_0^x t^{j+1} e^{\sigma(t)} d\sigma(t)
\]

\[
- \frac{\left( \int_0^x t^{j+1} e^{\sigma(t)} d\sigma(t) \right)^2}{\int_0^\infty t^j e^{\sigma(t)} d\sigma(t)}.
\]

Since

\[
\left( \int_0^\infty t^{j+1} e^{\sigma(t)} d\sigma(t) \right)^2 \leq \left( \int_0^x t^{j+2} e^{\sigma(t)} d\sigma(t) \right)^2 \leq \int_0^\infty t^{j+1} e^{\sigma(t)} d\sigma(t).
\]

Then
\[
G''(x) + \frac{G'(x)}{x} - \frac{(G'(x))^2}{G(x)} \geq \frac{1}{x} \int_{0}^{\infty} t e^{t x} d\sigma(t) \geq 0.
\]

The proof is complete.

It is known (see, for example [2, pp. 260-262]) that if \( f \) is exponentially convex, then
\[
f(x) = \int_{-\infty}^{\infty} e^{t x} d\sigma(t), \quad x \in (0, \infty),
\]
where \( \sigma \) is non-decreasing function on \( \mathbb{R} \). Moreover one has
\[
f^{(2k)}(x) = \int_{-\infty}^{\infty} t^{2k} e^{t x} d\sigma(t), \quad x \in (0, \infty), \quad k = 0, 1, 2, \ldots.
\]

**Corollary 3.5.** Let \( f \) be exponentially convex on \((0, \infty)\). Then for all \( k \in \mathbb{N} \) one has \( f^{(2k)}(x) \in (p, q) - Ho(0, \infty) \) for all \( p, q \) as above.

**Proof.** It is similar to Corollary 3.4.

**References**


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