
Bifurcation Solutions of Elastic Beams

Equation with Small Perturbation

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Abstract. In this paper bifurcation solution of elastic beams equation has been studied by using local method of Lyapunov-Schmidt. The bifurcation equation corresponding to the elastic beams equation has been found as a finite system of two equations. Also, the Discriminant set (bifurcation set) of the elastic beams equation has been found for some values of parameters.

Keywords: Bifurcation theory, Local Lyapunov-Schmidt method, Bifurcation set.

1. Introduction

It is known that many of the nonlinear problems that appear in Mathematics and Physics can be written in the form of operator equation,

\[ f(x, \lambda) = b, \quad x \in O \subset X, \quad b \in Y, \quad \lambda \in \mathbb{R}^n. \quad \ldots (1.1) \]

where \( f \) is a smooth Fredholm map of index zero and \( X, Y \) are Banach spaces and \( O \) is open subset of \( X \). For these problems, the method of reduction to finite dimensional equation,

\[ \Theta(\xi, \lambda) = \beta, \quad \xi \in M, \quad \beta \in N, \quad \ldots (1.2) \]

can be used, where \( M \) and \( N \) are smooth finite dimensional manifolds.

The reduction from equation (1.1) to the equation (1.2) (Variant local scheme of Lyapunov-Schmidt) with the conditions that equation (1.2) has all the topological and analytical properties of equation (1.1) (multiplicity, bifurcation diagram, etc) dealing with [3],[7],[11][12].

The oscillations and motion of waves of the elastic beams on elastic foundations can be described by means of the following PDE,
\[ \frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} + \alpha \frac{\partial^2 y}{\partial x^2} + \beta y + y^2 + y^3 = \psi, \]

where \( y \) is the deflection of beam and \( \psi = \varepsilon \varphi(x) \) (\( \varepsilon \) - small parameter) is a continuous function. It is known that, to study the oscillations of beams, equilibrium state \((w(x) = \gamma(x,t))\) should be considered which is describe by the equation,

\[ \frac{d^4 w}{d x^4} + \alpha \frac{d^2 w}{d x^2} + \beta w + w^2 + w^3 = \psi, \quad \ldots \quad (1.3) \]

equation (1.3) is a special case of the general nonlinear differential equation of fourth order,

\[ \frac{d^4 w}{d x^4} + \alpha \frac{d^2 w}{d x^2} + \beta w + g(\lambda, \tilde{w}) = \psi, \quad \ldots \quad (1.4) \]

When \( g(\lambda, \tilde{w}) = -k w^3 \) and \( \psi = \tilde{0} \), equation (1.4) has been studied as follows: Thompson and Stewart [4] showed numerically the existence of periodic solutions of equation (1.4) for some values of parameters. Bardin and Furta [1] used the local method of Lyapunov-Schmidt and found the sufficient conditions of existence of periodic waves of equation (1.4), also they are introduced the solutions of equation (1.4) in the form of power series. Furta and Piccione [8] showed the existence of periodic travelling wave solutions of equation (1.4) describing oscillations of an infinite beam, which lies on a non-linearly elastic support with non-small amplitudes. Sapronov [10] applied the local method of Lyapunov-Schmidt and found the bifurcation solutions of equation (1.4) when \( g(\lambda, \tilde{w}) = w^3 \) and \( \psi = \tilde{0} \) with the boundary conditions,

\[ w(0) = w(\pi) = w''(0) = w''(\pi) = 0. \]

in his study he solved the bifurcation equation corresponding to the equation (1.4) and found the bifurcation diagram of a specify problem. When \( g(\lambda, \tilde{w}) = w^2 \) and \( \psi \neq \tilde{0} \), equation (1.4) has been studied by Abdul Hussain [5], it was shown that by using local method of Lyapunov-Schmidt the existence of bifurcation solutions of equation (1.4) with the conditions,

\[ w(0) = w(1) = w''(0) = w''(1) = 0. \]

and another study with the conditions,

\[ w(x_1) \geq \varepsilon_1, \quad w(x_2) \geq \varepsilon_2, \]

\[ 0 < x_1 < x_2 < 1, \quad \varepsilon_1, \varepsilon_2 \text{ are small parameters.} \]

also, it was given a new study of corner singularities of smooth maps in the analysis of bifurcations balance of the elastic beams and periodic waves. When \( g(\lambda, \tilde{w}) = w^2 + w^3 \) equation (1.4) was studied by Sapronov [2], in his work he found bifurcation periodic solutions of equation (1.3) by using local method of Lyapunov-Schmidt. Also, he solved the bifurcation equation corresponding to
the equation (1.4) and found the bifurcation diagram of a specific problem. In this paper we used the local method of Lyapunov–Schmidt to study the bifurcation solutions of boundary value problem (the perturbation of equation (1.3)),

\[ \frac{d^4w}{dx^4} + \alpha \frac{d^2w}{dx^2} + (\beta + \epsilon_1 x) w + \epsilon_2 \frac{dw}{dx} + w^2 + w^3 = \psi, \]

\[ \ldots \quad (1.5) \]

where \( \epsilon_1 \) and \( \epsilon_2 \) are small parameters indicating the perturbation and \( \psi = \epsilon_1 \phi(x) \) (\( \epsilon \) – small parameter) is a symmetric function with respect to the involution \( I : \psi(x) \rightarrow \psi(x - \pi) \).

We shall introduce two basic definitions.

**Definition 1.1** Suppose that \( E \) and \( F \) are Banach spaces and \( A : E \rightarrow F \) be a linear continuous operator. The operator \( A \) is called Fredholm operator, if

1. The kernel of \( A \), \( \text{Ker}(A) \), is finite dimensional,
2. The range of \( A \), \( \text{Im}(A) \), is closed in \( F \),
3. The Cokernel of \( A \), \( \text{Coker}(A) \), is finite dimensional.

The number

\[ \dim(\text{Ker} A) - \dim(\text{Coker} A) \]

is called Fredholm index of the operator \( A \) and denote it by \( \text{ind}(A) \).

**Definition 1.2** The set of all \( \lambda \) in which equation (1.1) has degenerate solutions is called the Discriminant set.

### 2. Reduction to the bifurcation equation

To study the problem (1.5) it is convenient to set the ODE in the form of operator equation, that is,

\[ f(w, \lambda, \epsilon_1, \epsilon_2) = \frac{d^4w}{dx^4} + \alpha \frac{d^2w}{dx^2} + (\beta + \epsilon_1 x) w + \epsilon_2 \frac{dw}{dx} + w^2 + w^3 \]

\[ \ldots \quad (2.1) \]

where \( f : E \rightarrow F \) is a nonlinear Fredholm map of index zero from Banach space \( E \) to Banach space \( F \), \( E = C^4([0, \pi], R) \) is the space of all continuous functions that have derivative of order at most four, \( F = C^0([0, \pi], R) \) is the space of all continuous functions and \( w = w(x), \ x \in [0, \pi], \ \lambda = (\alpha, \beta) \). In this case the bifurcation solutions of equation (2.1) is equivalent to the bifurcation solutions of the operator equation,

\[ f(w, \lambda, \epsilon_1, \epsilon_2) = \psi, \quad \psi \in F . \]

\[ \ldots \quad (2.2) \]

It is clear that when \( \epsilon_1 \) and \( \epsilon_2 \) are both equal to zero, then the operator \( f \) have variational property that is; there exist a functional \( V : \Omega \rightarrow R \) such that \( f(w, \lambda, 0, 0) = \text{grad}_\lambda V(w, \lambda, 0) \) or equivalently,
\[
\frac{\partial V}{\partial w}(w, \lambda, 0) h = \langle f(w, \lambda, 0, 0), h \rangle_H, \quad \forall w \in \Omega, \ h \in E.
\]

where \( \langle \cdot, \cdot \rangle_H \) is the scalar product in Hilbert space \( H \) and \( V(w, \lambda, \psi) = \int_0^\pi \left( \frac{(w')^2}{2} - \alpha \frac{(w')^2}{2} + \beta \frac{w^2}{2} + \frac{w^3}{3} + \frac{w^4}{4} - w \psi \right) dx \).

In this case the solutions of the equation \( f(w, \lambda, 0, 0) = \psi \) are the critical points of the functional \( V(w, \lambda, \psi) \), where the critical points of the functional \( V(w, \lambda, \psi) \) are the solutions of Euler-Lagrange equation,

\[
\frac{\partial V}{\partial w}(w, \lambda, 0) h = \int_0^\pi \left( w'''' + \alpha w'' + \beta w + w^2 + w^3 \right) h dx = 0.
\]

The bifurcation solutions of problem (1.5) when \( \epsilon_1 = \epsilon_2 = 0 \) have been studied by Mohammed [6], in his work he showed that the Caustic of problem (1.5) is a union of two surfaces. Also, he showed the existence and stability solutions of a specify problem. If \( \epsilon_1 \) and \( \epsilon_2 \) are not both equal to zero, then the operator \( f \) should be lose the variational property, in this case we go to seek the existence of regular solutions of problem (1.5) in the plane of parameters by using local method of Lyapunov-Schmidt. It is well known that by finite dimensional reduction theorem the solutions of problem (1.5) is equivalent to the solutions of finite dimensional system with \( 2 = \dim(\ker f_w(0, \lambda)) \) variables and \( 2 = \dim(Co \ker f_w(0, \lambda)) \) equations, so first step in this work we shall find this system and then we analyze the results to find the bifurcation solutions of problem (1.5). Our purpose is to study the bifurcation solutions of problem (1.5) near the bifurcation solutions of the problem,

\[
\frac{d^4w}{dx^4} + \alpha \frac{d^2w}{dx^2} + \beta w + w^2 + w^3 = \psi,
\]

\[
w(0) = w(\pi) = w''(0) = w''(\pi) = 0.
\]

First step in this reduction determine the linearized equation corresponding to the equation (2.2) which is given by the following equation,

\[
A h = 0, \quad h \in E,
\]

\[
A = \frac{\partial f}{\partial w}(0, \lambda, 0, 0) = \frac{d^4}{dx^4} + \alpha \frac{d^2}{dx^2} + \beta,
\]

\[
h(0) = h(\pi) = h''(0) = h''(\pi) = 0.
\]

The solution of linearized equation satisfying the boundary conditions is given by,

\[
e_p(x) = c_p \sin(px), \quad p = 1, 2, 3, \ldots
\]

and the characteristic equation corresponding to this solution is,

\[
p^4 - \alpha p^2 + \beta = 0.
\]
This equation gives in the $\alpha\beta$-plane characteristic lines $\ell_p$. The characteristic lines $\ell_p$ consists the points $(\alpha, \beta)$ in which the linearized equation have nonzero solutions. The point of intersection of characteristic lines in the $\alpha\beta$-plane is a bifurcation point [10]. So for equation (2.2) the point $(\alpha, \beta) = (5, 4)$ is a bifurcation point. Localized parameters $\alpha, \beta$ as follows,

$$\alpha = 5 + \delta_1, \quad \beta = 4 + \delta_2, \quad \delta_1, \delta_2$$

are small.

lead to bifurcation along the modes $e_1(x) = c_1 \sin(x), \quad e_2(x) = c_2 \sin(2x)$, where $\|e_1\| = \|e_2\| = 1$, and $c_1 = c_2 = \sqrt{2/\pi}$.

Let $N = \text{Ker} A = \text{Span} \{e_1, e_2\}$, then the space $E$ can be decomposed in direct sum of two subspaces, $N$ and the orthogonal complement to $N$,

$$E = N \oplus E^{\times-2}, \quad E^{\times-2} = N^\perp \cap E = \{v \in E : v \perp N\}.$$

Similarly, the space $F$ can be decomposed in direct sum of two subspaces, $N$ and orthogonal complement to $N$,

$$F = N \oplus F^{\times-2}, \quad F^{\times-2} = N^\perp \cap F = \{f \in F : f \perp N\}$$

and hence every vector $w \in E$ can be written in the form,

$$w = u + v, \quad u = \sum_{i=1}^{2} \xi_i e_i \in N, \quad N \perp v \in E^{\times-2}, \quad \xi_i = \langle w, e_i \rangle.$$

Similarly,

$$f(w, \lambda, e_1, e_2) = f^{(2)}(w, \lambda, e_1, e_2) + f^{(\times-2)}(w, \lambda, e_1, e_2),$$

$$f^{(2)}(w, \lambda, e_1, e_2) = \sum_{i=1}^{2} v_i(w, \lambda, e_1, e_2) e_i \in N, \quad f^{(\times-2)}(w, \lambda, e_1, e_2) \in F^{\times-2},$$

$$v_i(w, \lambda, e_1, e_2) = \langle f(w, \lambda, e_1, e_2), e_i \rangle.$$
and then we have the bifurcation equation in the form,

$$\Theta(\xi, \lambda, \epsilon_1, \epsilon_2) = \psi_1, \quad \xi = (\xi_1, \xi_2), \quad \lambda = (\alpha, \beta),$$

where,

$$\Theta(\xi, \lambda, \epsilon_1, \epsilon_2) = f^{(2)}(u + \Phi(u, \lambda, \epsilon_1, \epsilon_2), \lambda, \epsilon_1, \epsilon_2).$$

Equation (2.1) can be written in the form,

$$f(u + v, \lambda, \epsilon_1, \epsilon_2) = A(u + v) + B(u + v) + T(u + v)$$

$$= Au + \epsilon_1 x u + \epsilon_2 u' + u^2 + u^3 +\ldots$$

where $B(u + v) = \epsilon_1 x (u + v) + \epsilon_2 (u + v)'$, $T(u + v) = (u + v)^2 + (u + v)^3$ and the dots denote the terms consists $u$ and $v$ together. Hence

$$\Theta(\xi, \lambda, \epsilon_1, \epsilon_2) = f^{(2)}(u + v, \lambda, \epsilon_1, \epsilon_2) = \sum_{i=1}^{2} \langle Au + \epsilon_1 x u + \epsilon_2 u' + u^2 + u^3, \epsilon_i \rangle e_i +\ldots = \psi_1 \quad \ldots (2.4)$$

Equation (2.4) implies that,

$$\sum_{i=1}^{2} \langle Au + \epsilon_1 x u + \epsilon_2 u' + u^2 + u^3, \epsilon_i \rangle e_i +\ldots = t_1 e_1 + t_2 e_2 \quad \ldots (2.5)$$

After some calculations of equation (2.5) we have the following result,

$$(A_1 \xi_1^3 + A_2 \xi_1 \xi_2^2 + A_3 \xi_2^2 + A_4 \xi_1 + A_5 \xi_1 + A_6 \xi_2) e_1 +$$

$$(B_1 \xi_2^3 + B_2 \xi_1 \xi_2^2 + B_3 \xi_2^2 + B_4 \xi_2 + B_5 \xi_1) e_2 = t_1 e_1 + t_2 e_2$$

where,

$$A_1 = B_1 = \frac{3}{2\pi}, \quad A_2 = B_2 = \frac{3}{\pi}, \quad A_3 = \frac{8}{3\pi} \sqrt{2}, \quad A_4 = \frac{4}{5},$$

$$B_3 = 2 A_4, \quad A_5 = \tilde{\alpha}_1(\lambda) + \frac{\epsilon_1 \pi}{2}, \quad A_6 = -\frac{16 \epsilon_1 + 24 \epsilon_2}{9\pi},$$

$$B_4 = \tilde{\alpha}_2(\lambda) + \frac{\epsilon_1 \pi}{2}, \quad B_5 = -\frac{16 \epsilon_1 - 24 \epsilon_2}{9\pi}$$

and $\tilde{\alpha}_1(\lambda), \tilde{\alpha}_2(\lambda)$ are spectral smooth functions. The symmetry of the function $\psi(x)$ with respect to the involution $I : \psi(x) \mapsto \psi(\pi - x)$ implies that $t_2 = 0$ and then we have state the following result,

**Theorem 2.1** The bifurcation equation

$$\Theta(\xi, \lambda, \epsilon_1, \epsilon_2) = f^{(2)}(u + \Phi(u, \lambda), \lambda, \epsilon_1, \epsilon_2) = \psi_1$$

corresponding to the equation (2.2) have the following form,
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\[ \Theta(\xi, \lambda) = \left( \xi_1^3 + 2\xi_1\xi_2^2 + 5b\xi_1^2 + 4b\xi_2^2 + \lambda_1\xi_1 + \lambda_2\xi_2 + q_1 \right) + O(\xi^3) = 0. \]

where,

\[ \xi = (\xi_1, \xi_2), \quad \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, q_1) \in \mathbb{R}^5, \quad \delta = (\delta_1, \delta_2), \quad b = \frac{16}{45}\sqrt{\frac{2}{\pi}}. \]

The Discriminant set \( \Sigma \) of the map \( \Theta(\xi, \lambda) \) is locally equivalent in the neighborhood of point zero to the Discriminant set of the map \( \Theta_1(\xi, \lambda) \) [9],

\[ \Theta_1(\xi, \lambda) = \left( \xi_1^3 + 2\xi_1\xi_2^2 + 5b\xi_1^2 + 4b\xi_2^2 + \lambda_1\xi_1 + \lambda_2\xi_2 + q_1 \right) \]

this mean that, to study the Discriminant set of the map \( \Theta(\xi, \lambda) \) it is sufficient to study the Discriminant set of the map \( \Theta_1(\xi, \lambda) \). The point \( a \in E \) is a solution of equation (2.2) if and only if

\[ a = \sum_{i=1}^{3} \overline{\xi}_i e_i + \Phi(\overline{\xi}, \lambda), \]

where \( \overline{\xi} \) is a solution of equation

\[ \Theta_1(\xi, \lambda) = 0. \]

also, the Discriminant set of equation (2.2) is equivalent to the Discriminant set of equation (2.7). In the complex variables \( z = \xi_1 + i\xi_2, \quad i = \sqrt{-1} \) equation (2.7) has the following form,

\[ -\frac{1}{4}z^3 + \frac{9}{4}z^2 - \frac{7}{4}bz^2 + \frac{9}{2}b|z|^2 + \gamma_1 z + \gamma_2 \overline{z} + q_1 = 0 \]

where, \( \overline{z} \) is the complex conjugate of the variable \( z \), \( |z| = \sqrt{\xi_1^2 + \xi_2^2} \) and

\[ \gamma_1 = \lambda_1 + i\lambda_3, \quad \gamma_2 = \lambda_2 + i\lambda_4, \quad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0, \]

\[ \lambda_3 = \frac{\lambda_1 - \lambda_2}{2}, \quad \lambda_4 = \frac{\lambda_1 + \lambda_2}{2}. \]

In polar coordinate system \( z = r e^{i\theta} \) equation (2.8) has the form,

\[ -\frac{1}{4}r^3 e^{-3i\theta} + \frac{9}{4}r^3 e^{i\theta} + \frac{9}{2}b r^2 e^{2i\theta} - \frac{7}{4}br^2 e^{-2i\theta} + \gamma_1 r e^{i\theta} + \gamma_2 r e^{-i\theta} + q_1 = 0 \]

We want to find the set of all \( \lambda_3 \) for which equation (2.7) have real degenerate solutions, so we shall take only the real part of the last equation to find this set. The real part of the last equation is given by the following equation,

\[ \alpha_1(\theta) r^3 + \alpha_2(\theta) r^2 + \alpha_3(\theta) r + q_1 = 0, \]

\[ \alpha_1(\theta) = \frac{\lambda_1 - \lambda_2}{2}, \quad \alpha_2(\theta) = \frac{\lambda_1 + \lambda_2}{2}. \]
where, \( \theta \) is given in radius and
\[
\begin{align*}
\alpha_1(\theta) &= \frac{5}{4} \cos \theta - \frac{1}{4} \cos 3\theta, \\
\alpha_2(\theta) &= b(4 + \cos^2 \theta), \\
\alpha_3(\theta) &= \lambda_1 \cos \theta + \lambda_2 \sin \theta
\end{align*}
\]

We interested with the Discriminant set of equation (2.9) only when \( \alpha_i(\theta) \neq 0 \), so the Discriminant set of equation (2.9) for \( \alpha_i(\theta) \neq 0 \) is given by the following equation
\[
\begin{align*}
-\frac{5}{972} \alpha_2^6 + \frac{73}{486} \alpha_1 \alpha_2^4 \alpha_3 + \frac{343}{48} \alpha_1^2 \alpha_2^2 \alpha_3^2 - \frac{363}{16} \alpha_1^3 \alpha_3^3 \\
- \frac{4}{27} \alpha_1 \alpha_3 q_1 - \alpha_1 q_1^2 + \frac{2}{3} \alpha_1^3 q_1 \alpha_2 \alpha_3 = 0.
\end{align*}
\]

To obtain the best section of Discriminant set in the plane of parameters \( q_1, \alpha_3 \) we choose \( \alpha_1 = 1, \ \alpha_2 = 5b \) and we describes the Discriminant set of equation (2.9) in the plane of parameters \( q_1, \alpha_3 \) by the following graph,

![Fig.1 describe the Discriminant set when \( \alpha_1 = 1, \ \alpha_2 = 5b \).](image)

In each region the number of regular solutions of equation (2.9) is determine to be one or three. From the solutions of equation (2.9) we can find the solutions of equation (2.2) by finding the values of \( \xi_1 \) and \( \xi_2 \). For example, if \( \alpha_3 = -35.5 \) and \( q_1 = 30 \) then equation (2.9) has the following three regular solutions \( r_1 = -7.053986245, \ r_2 = 0.8976398778 \) and \( r_3 = 4.737884926 \). The common value of \( \theta \) corresponding to these solutions is \( \theta = 0 \) and then the value of \( \lambda_1 \) must be equal to \( -35.5 \), so the values of \( \xi_1 \) are \( r_1, r_2 \) and \( r_3 \). The only value of \( \xi_2 \) is \( \xi_2 = 0 \). The solutions of equation (2.2) in the form \( w = \xi_1 \sin(x) + \xi_2 \sin(2x) \) for these values of \( \xi_1 \) and \( \xi_2 \) is given by the following figures,
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Fig. 2(a)\[\xi_1 = 0.8976398778, \xi_2 = 0.\]
\[\xi_1 = 4.737884926, \xi_2 = 0.\]

Fig. 2 (b)\[\xi_1 = -7.053986245, \xi_2 = 0.\]

Other solutions of equation (2.9) can be obtained by choosing different values of $\alpha_1, \alpha_2, \alpha_3$ and $q_1$. The solutions of equation (2.2) for the values of $\xi_1$ and $\xi_2$ of the form $w = \xi_1 \sin(x) + \xi_2 \sin(2x) + \Phi(u, \lambda, \varepsilon_1, \varepsilon_2)$ we will find in other paper by using the same procedure used in [1].

Conclusions

In the present work the bifurcation solutions of nonlinear elastic beams equation of fourth order was studied. The local method of Lyapunov-Schmidt has been used to find the bifurcation equation corresponding to the problem (1.5) with symmetry. This allowed us to find the Discriminant set of a specify problem as semi-cubic curve. The number of regular solutions of problem (1.5) has been found in some domain of parameters. In figure (2) we describe the solutions of problem (1.5) only for the linear part for some values of parameters and the variables $\xi_1, \xi_2$. In this work the Discriminant set of problem (1.5) is not completely determined.

References


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