Abstract. In this paper criterion for $L_1$- convergence of $N_n^{(2)}(x)$ cosine sums with quasi hyper convex coefficients is obtained.

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1. Introduction

In what follows we will denote by

(1)\hspace{1cm} g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,

with partial sums defined by

(2)\hspace{1cm} g_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx,

and

(3)\hspace{1cm} g(x) = \lim_{n \to \infty} g_n(x).

In the sequel we will mention some notations which are useful for the further work. Dirichlet’s kernels are denoted by

\[ D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin \left( n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}} \]

\[ \bar{D}_n(t) = \sum_{k=1}^{n} \cos kt \]

\[ \overline{\bar{D}}_n(t) = \sum_{k=1}^{n} \sin kt = \frac{\cos \frac{t}{2} - \cos \left( n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}} \]

\[ \overline{D}_n(t) = -\frac{1}{2} \cot \frac{t}{2} + \overline{\overline{D}}_n(t) = -\frac{\cos \left( n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}} \]
In what follows we will briefly describe some known facts which will be very useful for us (see [13]):

\[ S_n^0 = S_n = a_0 + a_1 + \cdots + a_n \]
\[ S_n^k = S_0^{k-1} + S_1^{k-1} + \cdots + S_n^{k-1}, \quad k = 1, 2, \cdots; n = 1, 2, \cdots; \]

(4) 
\[ A_n^0 = 1, \quad A_n^k = A_0^{k-1} + A_1^{k-1} + \cdots + A_n^{k-1}, \quad k = 1, 2, \cdots; n = 1, 2, \cdots; \]

The \( A_n \)'s are called the binomial coefficients and are given by the following relation:

(5) 
\[ \sum_{k=0}^{\infty} A_k^\alpha x^\alpha = (1 - x)^{(-\alpha - 1)}, \]

whereas \( S_n \)'s are given by

(6) 
\[ \sum_{k=0}^{\infty} S_k^\alpha x^\alpha = (1 - x)^{-\alpha} \sum_{k=0}^{\infty} S_k x^\alpha, \]

and

(7) 
\[ A_n^\alpha = \sum_{k=0}^{n} A_k^{\alpha-1}, \quad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}, \]

The Cesaro means \( T_k^\alpha \) of order \( \alpha \) is denoted by \( T_k^\alpha = \frac{S_k^\alpha}{A_k} \). Also for \( 0 < x \leq \pi \), let

\[ S_n^0(x) = \tilde{D}_n(x) = \cos x + \cos 2x + \cdots + \cos nx \]
\[ S_n^k(x) = S_0(x) + S_1(x) + \cdots + S_n(x), \]

(8) 
\[ S_n^k(x) = S_0^{k-1}(x) + S_1^{k-1}(x) + \cdots + S_n^{k-1}(x). \]

The Cesaro means \( T_k^\alpha(x) \) of order \( \alpha \) is denoted by \( T_k^\alpha(x) = \frac{S_k^\alpha(x)}{A_k} \).

**Lemma 1.1.** (see [2]) If \( \alpha \geq 0, p \geq 0, \epsilon_n = o(n^{-p}), \) and \( \sum_{n=0}^{\infty} A_n^{\alpha+p}|\Delta^{\alpha+1}\epsilon_n| < \infty, \) then

\[ \sum_{n=0}^{\infty} A_n^{\lambda+p}|\Delta^{\lambda+1}\epsilon_n| < \infty, \]

for \(-1 \leq \lambda \leq \alpha, A_n^{\lambda+p}\Delta^{\lambda}\epsilon_n \) is of bounded variation for \( 0 \leq \lambda \leq \alpha \) and tends to zero as \( n \to \infty \).

**Definition 1.2.** A sequence of scalars \( (a_n) \) is said to be twice quasi semi-convex if \( a_n \to 0 \) as \( n \to \infty \), and

(9) 
\[ \sum_{n=1}^{\infty} n|\Delta^4 a_{n-1} - \Delta^4 a_n| < \infty, \quad (a_0 = a_{-1} = 0), \]

where \( \Delta^4 a_n = \Delta^3 a_n - \Delta^3 a_{n+1} \).

The \( L_1 \)-convergence of cosine and sine sums was studied by several authors. Kolmogorov in [6], proved the following theorem:
Theorem 1.3. If \((a_n)\) is a quasi-convex null sequence, then for the \(L_1\)-convergence of the cosine series (1), it is necessary and sufficient that \(\lim_{n \to \infty} a_n \cdot \log n = 0\).

The case in which sequence \((a_n)\) is convex, of this theorem was established by Young (see [12]). That is why, sometimes, this theorem is known as Young-Kolmogorov Theorem.

Definition 1.4. A sequence of scalars \((a_n)\) is said to be twice quasi-convex if \(a_n \to 0\) as \(n \to \infty\), and

\[
\sum_{n=1}^{\infty} n|\Delta^4 a_{n-1}| < \infty, (a_0 = a_{-1} = 0), \tag{10}
\]

Definition 1.5. A sequence of scalars \((a_n)\) is said to be quasi hyper-convex if \(a_n \to 0\) as \(n \to \infty\), and

\[
\sum_{n=1}^{\infty} n^\alpha |\Delta^{\alpha+3} a_{n-1}| < \infty, (a_0 = a_{-1} = 0), \tag{11}
\]

for \(\alpha > 0\). For \(\alpha = 1\), this class reduces to the class defined in Definition 1.4.

Bala and Ram in [1] have proved that Theorem 1.3 holds true for cosine series with semi-convex null sequences in the following form:

Theorem 1.6. If \((a_n)\) is a semi-convex null sequence, then for the convergence of the cosine series (1) in the metric space \(L\), it is necessary and sufficient that \(a_k \log k = o(1), k \to \infty\).

Garret and Stanojevic in [4], have introduced modified cosine sums

\[
G_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_j) \cos kx. \tag{12}
\]

The same authors (see [5]), Ram in [10] and Singh and Sharma in [11] studied the \(L_1\)-convergence of this cosine sum under different sets of conditions on the coefficients \((a_n)\). Kumari and Ram in [9], introduced new modified cosine and sine sums as

\[
f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left( \frac{a_j}{j} \right) \cos kx, \tag{13}
\]

and

\[
G'_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta \left( \frac{a_j}{j} \right) \sin kx, \tag{14}
\]

and have studied their \(L_1\)-convergence under the condition that the coefficients \((a_n)\) belong to different classes of sequences. Later one, Kulwinder in [7], introduced new modified sine sums as

\[
K_n(x) = \frac{1}{2 \sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta a_{j-1} - \Delta a_{j+1}) \sin kx, \tag{15}
\]

and have studied their \(L_1\)-convergence under the condition that the coefficients \((a_n)\) are semi-convex null. In [8], was proved the \(L_1\) convergence of some modified cosine series using in consideration generalized semi-convex coefficients.
In this paper we call the modified cosine sums defined in [3] as follows:

\( N^{(2)}_n(x) = \frac{1}{(2 \sin \frac{x}{2})^4} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta^4 a_{j-2} - \Delta^4 a_{j-1}) \cos kx + \frac{a_1 (\cos x - 4)}{(2 \sin \frac{x}{2})^4} + \frac{a_2}{(2 \sin \frac{x}{2})^4}, \)

and we will prove that this sums \( L_1 \)-converges to \( g(x) \), under conditions that coefficients \( (a_n) \) are quasi hyper-convex. In paper [3], was proved that the above modified cosine sums \( L_1 \)-converges to \( g(x) \) under condition that coefficients \( (a_n) \) are twice quasi semi-convex and is proved also that necessary and sufficient condition for \( L_1 \)-convergence of the cosine series (1) is \( \lim_{n \to \infty} a_n \log n = 0 \).

First we will prove this trivial fact:

**Lemma 2.1.** If \( (a_n) \) is a quasi hyper-convex null sequence of scalars, then it follows that the following relation

\( \sum_{k=1}^{\infty} k^\alpha |(\Delta^{\alpha+3} a_{k-1} - \Delta^{\alpha+3} a_k)| < \infty \)

holds.

**Proof.** The proof of the lemma follows directly from the relation (11):

\( \sum_{k=1}^{\infty} k^\alpha |(\Delta^{\alpha+3} a_{k-1} - \Delta^{\alpha+3} a_k)| \leq \sum_{k=1}^{\infty} k^\alpha |\Delta^{\alpha+3} a_{k-1}| + \sum_{k=1}^{\infty} k^\alpha |\Delta^{\alpha+3} a_k| < \infty. \)

**Lemma 2.2.** If \( (a_n) \) is a quasi hyper-convex null sequence of scalars, then it is twice quasi semi-convex null sequence too.

**Proof.** Because \( (a_n) \) is a quasi hyper-convex, then it follows that relation (17), from lemma 2.1, holds for every \( \alpha > 0 \), and for \( \alpha = 1 \), too. From this we get the following relation

\( \sum_{n=1}^{\infty} n |\Delta^{4} a_{n-1} - \Delta^{4} a_n| < \infty. \)

The next Theorem, also appears in [3]. For the convenience of reader we give its proof.

**Theorem 2.3.** Let \( (a_n) \) be a twice quasi semi-convex null sequence, then \( N^{(2)}_n(x) \) converges to \( g(x) \) in \( L_1 \) norm.

**Proof.** We have

\[ S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cdot \cos kx = \frac{1}{(2 \sin \frac{x}{2})^4} \cdot \sum_{k=1}^{n} a_k \cdot \cos kx \cdot \left(2 \sin \frac{x}{2}\right)^4 \]

\[ = \frac{1}{(2 \sin \frac{x}{2})^4} \cdot \sum_{k=1}^{n} a_k [\cos (k+2)x - 4 \cos (k+1)x + 6 \cos kx - 4 \cos (k-1)x + \cos (k-2)x] \]
Also, since \( \Delta \) = \( \lim \) for every \( \epsilon > 0 \),

\[
g(x) = \lim_{n \to \infty} S_n(x) = \frac{1}{(2 \sin \frac{x}{2})^4} \cdot \sum_{k=1}^{n-1} \left( \Delta^4 a_{k-2} - \Delta^4 a_{k-1} \right) \tilde{D}_k(x) + \frac{a_1 (\cos x - 4)}{(2 \sin \frac{x}{2})^4} + \frac{a_2}{(2 \sin \frac{x}{2})^4}
\]

Also

\[
N_n^{(2)}(x) = \frac{1}{(2 \sin \frac{x}{2})^4} \sum_{k=1}^{n} \sum_{j=k}^{n} \left( \Delta^4 a_{j-2} - \Delta^4 a_{j-1} \right) \cos kx + \frac{a_1 (\cos x - 4)}{(2 \sin \frac{x}{2})^4} + \frac{a_2}{(2 \sin \frac{x}{2})^4}
\]

respectively

\[
N_n^{(2)}(x) = \frac{1}{(2 \sin \frac{x}{2})^4} \sum_{k=1}^{n} \Delta^4 a_{k-2} \cos kx - \Delta^4 a_{n-1} \cdot \tilde{D}_n(x) \frac{a_1 (\cos x - 4)}{(2 \sin \frac{x}{2})^4} + \frac{a_2}{(2 \sin \frac{x}{2})^4}
\]

Now applying Abel’s transformation we get the following relation:

\[
N_n^{(2)}(x) = \frac{1}{(2 \sin \frac{x}{2})^4} \sum_{k=1}^{n-1} \left( \Delta^4 a_{k-2} - \Delta^4 a_{k-1} \right) \tilde{D}_k(x) - \frac{\Delta^4 a_{n-2} \cdot \tilde{D}_n(x)}{(2 \sin \frac{x}{2})^4} - \frac{\Delta^4 a_{n-1} \cdot \tilde{D}_n(x)}{(2 \sin \frac{x}{2})^4} + \frac{a_1 (\cos x - 4)}{(2 \sin \frac{x}{2})^4} + \frac{a_2}{(2 \sin \frac{x}{2})^4}.
\]

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Thus, we have
\[ g(x) - N_n^{(2)}(x) = \frac{1}{(2 \sin \frac{x}{2})^4} \sum_{k=n+1}^{\infty} (\Delta^4 a_{k-2} - \Delta^4 a_{k-1}) \tilde{D}_k(x) + \frac{\Delta^4 a_{n-2} \cdot \tilde{D}_n(x)}{(2 \sin \frac{x}{2})^4} + \frac{\Delta^4 a_{n-1} \cdot \tilde{D}_n(x)}{(2 \sin \frac{x}{2})^4} \Rightarrow \]

\[ g(x) - N_n^{(2)}(x) = \lim_{m \to \infty} \left( \frac{1}{(2 \sin \frac{x}{2})^4} \sum_{k=n+1}^{m} (\Delta^4 a_{k-2} - \Delta^4 a_{k-1}) \tilde{D}_k(x) \right) + \frac{\Delta^4 a_{n-2} \cdot \tilde{D}_n(x)}{(2 \sin \frac{x}{2})^4} + \frac{\Delta^4 a_{n-1} \cdot \tilde{D}_n(x)}{(2 \sin \frac{x}{2})^4}. \]

Thus, we have
\[ \int_0^n |g(x) - N_n^{(2)}(x)| \, dx \to 0, \]
for \( n \to \infty \).

**Theorem 2.4.** Let \((a_n)\) be a quasi hyper-convex null sequence, then \(N_n^{(2)}(x)\) converges to \(g(x)\) in \(L_1\) norm.

**Proof.** Let us start from the modified cosine sums:
\[ N_n^{(2)}(x) = \frac{1}{(2 \sin \frac{x}{2})^4} \sum_{k=1}^{n} \sum_{j=k}^{n} (\Delta^4 a_{j-2} - \Delta^4 a_{j-1}) \cos kx + \frac{a_1(\cos x - 4)}{(2 \sin \frac{x}{2})^4} + \frac{a_2}{(2 \sin \frac{x}{2})^4}. \]

From Theorem 2.3, it follows that

\[ ||g(x) - N_n^{(2)}(x)||_{L_1} \to 0, \quad n \to \infty, \]

if \((a_n)\), are twice quasi semi-convex null coefficients (in our case sequence \((a_n)\), is twice quasi semi-convex (see Lemma 2.2)). In what follows we will prove that

\[ ||g(x) - N_n^{(2)}(x)||_{L_1} \to 0, \quad n \to \infty, \]

if \((a_n)\), are quasi hyper-convex null coefficients, using in consideration Cesaro’s mean of integral order. Applying Abel’s transformation, we have

\[ N_n^{(2)}(x) = \frac{1}{(2 \sin \frac{x}{2})^4} \sum_{k=1}^{n-1} (\Delta^4 a_{k-2} - \Delta^4 a_{k-1}) \tilde{D}_k(x) - \frac{\Delta^4 a_{n-2} \cdot \tilde{D}_n(x)}{(2 \sin \frac{x}{2})^4} - \frac{\Delta^4 a_{n-1} \cdot \tilde{D}_n(x)}{(2 \sin \frac{x}{2})^4} + \frac{a_1(\cos x - 4)}{(2 \sin \frac{x}{2})^4} + \frac{a_2}{(2 \sin \frac{x}{2})^4}. \]

If we use Abel’s transformation \(\alpha\) times, we get this relation:

\[ N_n^{(2)}(x) = \frac{1}{(2 \sin \frac{x}{2})^4} \sum_{k=1}^{n-\alpha} (\Delta^{\alpha+3} a_{k-2} - \Delta^{\alpha+3} a_{k-1}) S_k^{\alpha-2}(x) - \sum_{k=1}^{\alpha-1} \frac{(\Delta^{k+3} a_{n-k-2} - \Delta^{k+3} a_{n-k-1}) S_{n-k}^{k+1}(x)}{(2 \sin \frac{x}{2})^2} \]

\[ \frac{\Delta^4 a_{n-2} + \Delta^4 a_{n-1}}{(2 \sin \frac{x}{2})^4} \tilde{D}_n(x) + \frac{a_1(\cos x - 4)}{(2 \sin \frac{x}{2})^4} + \frac{a_2}{(2 \sin \frac{x}{2})^4}. \]

Since \(S_k^n(x), \, T_n(x), \, \tilde{D}_n(x)\) are uniformly bounded in any segment \([\epsilon, \pi - \epsilon]\), for any \(\epsilon > 0\), and \(T_k^n = \frac{S_k^n(x)}{A_k^n}\) we have

\[ g(x) = \lim_{n \to \infty} N_n^{(2)}(x) = \frac{1}{(2 \sin \frac{x}{2})^4} \sum_{k=1}^{\infty} (\Delta^{\alpha+3} a_{k-2} - \Delta^{\alpha+3} a_{k-1}) S_k^{\alpha-2}(x) + \frac{a_1(\cos x - 4)}{(2 \sin \frac{x}{2})^4} + \frac{a_2}{(2 \sin \frac{x}{2})^4}. \]
From relations (18) and (19) we have:

\[
g(x) - N_n^{(2)}(x) = \frac{1}{(2 \sin \frac{\pi}{2})^4} \sum_{k=-\infty}^{\infty} (\Delta^{\alpha+3} a_{k} - \Delta^{\alpha+3} a_{k-1}) S_n^{x_2}(x) + \\
\sum_{k=1}^{\alpha-1} \frac{(\Delta^{k+3} a_{n-k-2} - \Delta^{k+3} a_{n-k-1}) S_n^{k-1}(x)}{(2 \sin \frac{\pi}{2})^4} + \frac{\Delta^{4} a_{n-2} + \Delta^{4} a_{n-1} \bar{D}_n(x)}{(2 \sin \frac{\pi}{2})^4}.
\]

Respectively

\[
\|g(x) - N_n^{(2)}(x)\| \leq \left| \frac{1}{(2 \sin \frac{\pi}{2})^4} \sum_{k=-\infty}^{\infty} (\Delta^{\alpha+3} a_{k} - \Delta^{\alpha+3} a_{k-1}) S_n^{x_2}(x) \right| + \\
\left| \frac{1}{(2 \sin \frac{\pi}{2})^4} \sum_{k=1}^{\alpha-1} \Delta^{k+3} a_{n-k-2} S_n^{k-1}(x) \right| + \left| \frac{1}{(2 \sin \frac{\pi}{2})^4} \sum_{k=1}^{\alpha-1} \Delta^{k+3} a_{n-k-1} S_n^{k-1}(x) \right| + \left| \frac{\Delta^{4} a_{n-2} \bar{D}_n(x)}{(2 \sin \frac{\pi}{2})^4} \right|
\]

Finally we will have

\[
\|g(x) - N_n^{(2)}(x)\| \leq C_1 \int_0^\pi \left| \sum_{k=-\infty}^{\infty} (\Delta^{\alpha+3} a_{k} - \Delta^{\alpha+3} a_{k-1}) S_n^{x_2}(x) \right| dx + \\
C_1 \int_0^\pi \left| \sum_{k=1}^{\alpha-1} \Delta^{k+3} a_{n-k-2} S_n^{k-1}(x) \right| dx + C_1 \int_0^\pi \left| \sum_{k=1}^{\alpha-1} \Delta^{k+3} a_{n-k-1} S_n^{k-1}(x) \right| dx + \\
C_1 \int_0^\pi \left| \Delta^{4} a_{n-2} \bar{D}_n(x) \right| dx + C_1 \int_0^\pi \left| \Delta^{4} a_{n-1} \bar{D}_n(x) \right| dx
\]

\[
\leq C_1 \cdot \sum_{k=-\infty}^{\infty} A_k^{\alpha-2} \left| (\Delta^{\alpha+3} a_{k} - \Delta^{\alpha+3} a_{k-1}) \right| \int_0^\pi \left| T_n^{\alpha-2}(x) \right| dx + \\
C_1 \cdot \sum_{k=1}^{\alpha-1} A_k^{\alpha-2} \left| \Delta^{k+3} a_{n-k-2} \right| \int_0^\pi \left| T_n^{k-1}(x) \right| dx + C_1 \cdot \sum_{k=1}^{\alpha-1} A_k^{\alpha-2} \left| \Delta^{k+3} a_{n-k-1} \right| \int_0^\pi \left| T_n^{k-1}(x) \right| dx + \\
C_1 \cdot A_0^{\alpha} \cdot |\Delta^{4} a_{n-2}| \int_0^\pi |T_n^{\alpha}(x)| dx + C_1 \cdot A_0^{\alpha} \cdot |\Delta^{4} a_{n-1}| \int_0^\pi |T_n^{\alpha}(x)| dx.
\]

From lemma 1.1 and lemma 2.1 we get \( \|g(x) - N_n^{(2)}(x)\| \to 0 \), where \( n \to \infty \). \( \square \)
References


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