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Finite Difference Formulae for Unequal Sub-Intervals Using Lagrange's Interpolation Formula

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Abstract

General finite difference formulae and the corresponding error terms have been derived considering unequally spaced grid points, and using Lagrange's interpolation formula. Further the finite difference formulae and the error terms for equally spaced sub-intervals have also been obtained as special cases of the present study.

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1. Introduction

Finite difference method is one of the very effective methods used for solving the differential equations (ordinary or partial) numerically. It involves replacing the derivatives appearing in the differential equation and boundary conditions by suitable finite difference approximations. The accuracy of the solution depends upon the number of grid points, chosen. By increasing the number of grid points one can increase the accuracy of the solution to a desire degree, however it involves increasingly tedious mathematical analysis.

Based on Tayor's series, Khan and Ohba [2-8] presented some new difference schemes for finite difference approximation. They obtained closedforms expressions of these new difference formulae, which can give approximations of arbitrary order. Recently, using Lagrange's interpolation formula, Singh and Thorpe [9] have given a general method from which various types of finite difference formulae can be obtained by assigning the suitable values to the parameters. Further the method also facilitates the generation of finite difference formulae for higher derivatives by differentiation. However the applicability of the above methods appears to be limited as their method holds only when the grid points are equally spaced.

Another class of problem concerned with the finite difference formulae in the numerical analysis is to find them in case of unequal subintervals. This situation specially arises in the mathematical modelling when the function has been obtained experimentally and the independent variable is not under the control of the experimenter, thus one need to find the finite difference formulae for the observational data. To tackle this situation, by introducing generalized Vandermonde determinant, Li [2] presented general explicit difference formulae with arbitrary order accuracy for approximating first and higher order derivatives, which can be used for both equally and unequally spaced data. However, we use the available Lagrange's interpolation formula to obtain the finite difference formulae for unequally spaced subintervals.

2. Analysis

Using the Lagrange's interpolation formula, function $f(x)$ can be expressed as[1]

$$f(x) = \sum_{j=0}^{n} l_j(x) f_j \tag{1}$$

where f_j stands for $f(x_j)$, while $l_j(x)$ is as given below

$$l_j(x) = \frac{\pi(x)}{(x - x_j)\pi'(x_j)}. \tag{2}$$

In the above equation (2), prime denotes differentiation with respect to x and

$$\pi(x) = (x - x_0)(x - x_1)\,.\,.\,.\,.\,(x - x_n). \tag{3}$$

The truncation error, in the evaluation of $f(x)$ is as given below:

$$E_n(x) = \frac{\pi(x)}{(n+1)!} f^{\,n+1}(\xi) \tag{4}$$

where $f^{\,n+1}(\xi)$ denotes $(n+1)^{th}$ derivative of $f(\xi)$, while ξ lies between the interval $[x_0, x_n]$. The interval $[x_0, x_n]$ be divided into n subintervals of unequal widths $h_1, \; h_2, \; h_3, \; . \; . \; ., h_n$ such that $x_n = x_0 + \sum_{i=1}^{n} h_i$.

(a) Three Point Finite difference formulae: For this case $n = 2$, and hence setting

$$x - x_0 = (s+1)h_1, \; x - x_1 = sh_1 \text{ and } x - x_2 = sh_1 - h_2 \tag{5}$$

in equation (1) we have expression for $f(x)$ as

$$f(x) = \frac{s(sh_1 - h_2)}{(h_1 + h_2)} f(x_0) - \frac{(s+1)(sh_1 - h_2)}{h_2} f(x_1) + \frac{s(s+1)h_1^2}{h_2(h_1 + h_2)} f(x_2). \tag{6}$$

Differentiating equation (6) with respect to x, we get

$$f'(x) = \frac{(2sh_1 - h_2)}{h_1(h_1 + h_2)} f_0 - \frac{(2s+1)h_1 - h_2}{h_1 h_2} f_1 + \frac{(2s+1)h_1}{h_2(h_1 + h_2)} f_2. \tag{7}$$

Then by putting respectively, $s = -1, \; 0$ and h_2 / h_1, the formulae for the first order derivatives of $f(x)$ at the points $x_0, \; x_1, \; x_2$ can be obtained as

$$f'(x_0) = -\frac{(2h_1 + h_2)}{h_1(h_1 + h_2)} f_0 + \frac{h_1 + h_2}{h_1 h_2} f_1 - \frac{h_1}{(h_1 + h_2)h_2} f_2, \tag{8a}$$

$$f'(x_1) = -\frac{h_2}{h_1(h_1 + h_2)} f_0 - \frac{h_1 - h_2}{h_1 h_2} f_1 + \frac{h_1}{h_2(h_1 + h_2)} f_2, \tag{8b}$$

and

$$f'(x_2) = \frac{h_2}{h_1(h_1 + h_2)} f_0 - \frac{h_1 + h_2}{h_1 h_2} f_1 + \frac{h_1 + 2h_2}{h_2(h_1 + h_2)} f_2. \tag{8c}$$

Using (5) in equation (4), the error terms have been calculated as follows

$$E_2 = \frac{s(s+1)(sh_1 - h_2)h_2^2}{6} f'''(\xi) \tag{9}$$

and

$$E_2' = \frac{(3s^2 h_1 - 2sh_2 + 2sh_1 - h_2)h_1}{6} f'''(\xi). \tag{10}$$

The corresponding error terms in the formulae (8a, b, c) for $s = -1, 0$ and h_2 / h_1, result in respectively as $\frac{h_1(h_1 + h_2)}{6} f'''(\xi)$, $-\frac{h_1 h_2}{6} f'''(\xi)$ and $\frac{(h_1 + h_2)h_2}{6} f'''(\xi)$. Differentiation of equation (7) produces difference formula for second derivative as

$$f''(x) = \frac{2\left[h_2 f_0 - (h_1 + h_2) f_1 + h_1 f_2\right]}{h_1 h_2 (h_1 + h_2)}, \tag{11}$$

and the corresponding error term can be obtained by differentiating equation (10). For $h = h_1 = h_2$, we obtain the finite difference formulae and the corresponding truncation errors for equal sub-intervals as obtained by Singh and Thorpe [9].

(b) Four point finite difference formulae: In this case $n = 3$, and substituting (Singh and Thorpe[10])

$$x - x_0 = (s+1)h_1,\ x - x_1 = sh_1,\ x - x_2 = sh_1 - h_2 \text{ and } x - x_3 = sh_1 - (h_2 + h_3) \tag{12}$$

in the corresponding equation of (1), we get

$$f(x) = \frac{s(sh_1 - h_2)(h_2 + h_3 - sh_1)}{(h_1 + h_2)H_1} f_0 + \frac{(s+1)(sh_1 - h_2)(sh_1 - h_2 - h_3)}{h_2(h_2 + h_3)} f_1$$

$$+ \frac{s(s+1)h_1^2(h_2 + h_3 - sh_1)}{(h_1 + h_2)h_2 h_3} f_2 + \frac{s(s+1)h_1^2(sh_1 - h_2)}{H_1(h_2 + h_3)h_3} f_3 \tag{13}$$

where $H_1 = h_1 + h_2 + h_3$. Differentiation of (13) produces

$$f'(x)=\frac{(2sh_1-h_2)(h_2+h_3)+(2h_2-3sh_1)sh_1}{h_1(h_1+h_2)H_1}f_0$$

$$+\frac{(sh_1-h_2)(3sh_1+2h_1-h_2-h_3)-(s+1)h_1h_3}{h_1h_2(h_1+h_3)}f_1$$

$$+\frac{(2s+1)h_1(h_2+h_3)-s(3s+2)h_1^2}{(h_1+h_2)h_2h_3}f_2+\frac{(3s+2)sh_1^2-(2s+1)h_1h_2}{H_1(h_2+h_3)h_3}f_3. \quad (14)$$

For $s=-1,\ 0,\ h_2/h_1$ and $(h_2+h_3)/h_1$, the respective finite difference formulae are obtained from (14) as follows:

$$f'(x_0)=-\frac{(2h_1+h_2)H_1+h_1(h_1+h_2)}{h_1(h_1+h_2)H_1}f_0+\frac{(h_1+h_2)H_1}{h_1h_2(h_1+h_3)}f_1$$

$$-\frac{h_1H_1}{(h_1+h_2)h_2h_3}f_2+\frac{h_1(h_1+h_2)}{H_1(h_2+h_3)h_3}f_3, \quad (15a)$$

$$f'(x_1)=-\frac{h_2(h_2+h_3)}{h_1(h_1+h_2)H_1}f_0+\frac{h_2(h_2+h_3)-h_1(2h_2+h_3)}{h_1h_2(h_1+h_3)}f_1$$

$$-\frac{h_1(h_2+h_3)}{(h_1+h_2)h_2h_3}f_2-\frac{h_1h_2}{H_1(h_2+h_3)h_3}f_3, \quad (15b)$$

$$f'(x_2)=\frac{h_2h_3}{h_1(h_1+h_2)H_1}f_0-\frac{(h_1+h_2)h_3}{h_1h_2(h_1+h_3)}f_1$$

$$+\frac{h_3(h_1+2h_2)-h_2(h_1+h_2)}{(h_1+h_2)h_2h_3}f_2+\frac{(h_1+h_2)h_2}{H_1(h_2+h_3)h_3}f_3, \quad (15c)$$

$$f'(x_3)=-\frac{(h_2+h_3)h_3}{h_1(h_1+h_2)H_1}f_0+\frac{H_1h_3}{h_1h_2(h_1+h_3)}f_1$$

$$-\frac{H_1(h_2+h_3)}{(h_1+h_2)h_2h_3}f_2+\frac{(3H_1-h_1)h_3+H_1h_2}{H_1(h_2+h_3)h_3}f_3. \quad (15d)$$

The associated error term for four point formula is obtained from equation (4) as

$$E_3' = \frac{h_1}{24}\left[\left\{\left(3s^2 + 2s\right)h_1 - \left(2s+1\right)h_2\right\}\left(sh_1 - h_2 - h_3\right) + h_1\left(s^2+s\right)\left(sh_1 - h_2\right)\right] f^{iv}\left(\xi\right). \quad (16)$$

The truncation errors corresponding to equations (15a, b, c, d) are derived respectively, from equation (16) as
$\left[-H_1 h_1\left(h_1+h_2\right),\ h_1h_2\left(h_2+h_3\right),\ -h_2h_3\left(h_1+h_2\right),\ H_1h_3\left(h_2+h_3\right)\right]\ f^{iv}\left(\xi\right)/24$.
Differentiation of equation (14) gives difference formulae corresponding to the second derivatives as

$$f''(x) = \frac{\left(4h_2 + 2h_3 - 6sh_1\right)}{h_1\left(h_1+h_2\right)H_1} f_0 + \frac{2\left(3sh_1 + h_1 - 2h_2 - h_3\right)}{h_1h_2\left(h_1+h_3\right)} f_1 + \frac{2\left(h_2 + h_3 - h_1 - 3sh_1\right)}{\left(h_1+h_2\right)h_2h_3} f_2 + \frac{2\left(3sh_1 + h_1 - h_2\right)}{H_1\left(h_2+h_3\right)h_3} f_3 . \quad (17)$$

For $s = -1,\ 0,\ h_2/h_1$ and $\left(h_2+h_3\right)/h_1$, expression (17) results in

$$f''(x_0) = \frac{2\left(3h_1 + 2h_2 + h_3\right)}{h_1\left(h_1+h_2\right)H_1} f_0 - \frac{2\left(2h_1 + 2h_2 + h_3\right)}{h_1h_2\left(h_1+h_3\right)} f_1 + \frac{2\left(h_1 + H_1\right)}{\left(h_1+h_2\right)h_2h_3} f_2 - \frac{2\left(2h_1 + h_2\right)}{H_1\left(h_2+h_3\right)h_3} f_3 , \quad (18a)$$

$$f''(x_1) = \frac{2\left(2h_2 + h_3\right)}{h_1\left(h_1+h_2\right)H_1} f_0 + \frac{2\left(h_1 - 2h_2 - h_3\right)}{h_1h_2\left(h_1+h_3\right)} f_1 + \frac{2\left(H_1 - 2h_1\right)}{\left(h_1+h_2\right)h_2h_3} f_2 + \frac{2\left(h_1 - h_2\right)}{H_1\left(h_2+h_3\right)h_3} f_3 , \quad (18b)$$

$$f''(x_2) = \frac{2\left(h_3 - h_2\right)}{h_1\left(h_1+h_2\right)H_1} f_0 + \frac{2\left(H_1 - 2h_3\right)}{h_1h_2\left(h_1+h_3\right)} f_1 + \frac{2\left(h_3 - 2h_2 - h_1\right)}{\left(h_1+h_2\right)h_2h_3} f_2 + \frac{2\left(h_1 + 2h_2\right)}{H_1\left(h_2+h_3\right)h_3} f_3 , \quad (18c)$$

and
$$f''(x_3) = \frac{-2\left(h_2 + 2h_3\right)}{h_1\left(h_1+h_2\right)H_1} f_0 + \frac{2\left(H_1 + h_3\right)}{h_1h_2\left(h_1+h_3\right)} f_1$$

$$-\frac{2(2H_1-h_1)}{(h_1+h_2)h_2h_3}f_2+\frac{2(h_1+2h_2+3h_3)}{H_1(h_2+h_3)h_3}f_3, \tag{18d}$$

respectively. Truncation error concerned with equation (17) is obtained from differentiation of equation (16) as

$$E_3''=\frac{1}{12}\left[(sh_1-h_2)(5sh_1+2h_1-h_2-h_3)+h_1\left\{(s^2+s)h_1-(2s+1)h_3\right\}\right]f^{iv}(\xi), \tag{19}$$

from which respective truncation errors corresponding to equations (18) can be derived. Differentiation of equation (17) gives

$$f'''(x)=\frac{6}{h_1}\left[-\frac{1}{h_1(h_1+h_2)H_1}f_0+\frac{1}{h_1h_2(h_1+h_3)}f_1 -\frac{1}{(h_1+h_2)h_2h_3}f_2+\frac{1}{H_1(h_2+h_3)h_3}f_3\right] \tag{20}$$

which is the difference formula for the third derivative in terms of the function at the four points situated at unequal intervals. The corresponding truncation error can be obtained by differentiating equation (19). Putting $h=h_1=h_2=h_3$ in above expressions, we obtained the results for equal sub-intervals as obtained by Singh and Thorpe [9].

(c) Five point finite difference formulae: The value of n is four for five point difference formulae. Substituting

$$x-x_0=(s+1)h_1,\ x-x_1=sh_1,\ x-x_2=sh_1-h_2,$$
$$x-x_3=sh_1-(h_2+h_3),\ x-x_4=sh_1-(h_2+h_3+h_4) \tag{21}$$

in the corresponding equation obtained from (1), and then differentiating the resulting equation, we get

$$f'(x)=\frac{(2sh_1-h_2)(sh_1-h_2-h_3)(sh_1-H_2+h_1)+sh_1(sh_1-h_2)(2sh_1-2h_2-2h_3-h_4)}{h_1(h_1+h_2)(h_1+h_2+h_3)(h_1+h_2+h_3+h_4)}f_0$$
$$-\frac{(2sh_1+h_1-h_2)(sh_1-h_2-h_3)(sh_1-H_2+h_1)+h_1(s+1)(sh_1-h_2)(2sh_1-2h_2-2h_3-h_4)}{h_1h_2(h_1+h_3)(h_2+h_3+h_4)}f_1$$
$$+\frac{(2sh_1+h_1)(sh_1-h_2-h_3)(sh_1-H_2+h_1)+h_1^2(s^2+s)(2sh_1-2h_2-2h_3-h_4)}{(h_1+h_2)h_2h_3(h_3+h_4)}f_2$$

$$-\frac{(2sh_1+h_1)(sh_1-h_2)(sh_1-H_2+h_1)+h_1^2(s^2+s)(2sh_1-2h_2-h_3-h_4)}{(h_1+h_2+h_3)(h_2+h_3)h_3h_4}f_3$$

$$+\frac{(2s+1)h_1(sh_1-h_2)(sh_1-h_2-h_3)+h_1^2(s^2+s)(2sh_1-2h_2-h_3)}{(h_1+h_2+h_3+h_4)(h_2+h_3+h_4)(h_3+h_4)h_4}f_4 \qquad (22)$$

The expressions for the above derivative at $x=x_0$, x_2 and x_4 are obtained by setting respectively, $s=-1$, h_2/h_1 and $(h_2+h_3+h_4)/h_1$:

$$f'(x_0)=-\frac{(2h_1+h_2)H_1H_2+h_1(h_1+h_2)(2H_2-h_4)}{h_1(h_1+h_2)H_1H_2}f_0+\frac{(h_1+h_2)H_1H_2}{h_1h_2(h_1+h_3)(H_2-h_1)}f_1$$

$$-\frac{h_1H_1H_2}{(h_1+h_2)h_2h_3(h_3+h_4)}f_2+\frac{h_1(h_1+h_2)H_2}{H_1(h_2+h_3)h_3h_4}f_3-\frac{h_1(h_1+h_2)H_1}{H_2(H_2-h_1)(h_3+h_4)h_4}f_4 \qquad (23)$$

$$f'(x_2)=\frac{h_2h_3(h_3+h_4)}{h_1(h_1+h_2)H_1H_2}f_0-\frac{(h_1+h_2)h_3(h_3+h_4)}{h_1h_2(h_1+h_3)(H_2-h_1)}f_1$$

$$+\frac{(h_1+2h_2)h_3(h_3+h_4)-(h_1+h_2)h_2(2h_3+h_4)}{(h_1+h_2)h_2h_3(h_3+h_4)}f_2$$

$$+\frac{(h_1+h_2)h_2(h_3+h_4)}{H_1(h_2+h_3)h_3h_4}f_3-\frac{h_1(h_1+h_2)H_1}{H_2(H_2-h_1)(h_3+h_4)h_4}f_4 \qquad (24)$$

and

$$f'(x_4)=\frac{(H_2-h_1)(h_3+h_4)h_4}{h_1(h_1+h_2)H_1H_2}f_0-\frac{H_2(h_3+h_4)h_4}{h_1h_2(h_1+h_3)(H_2-h_1)}f_1$$

$$+\frac{H_2(H_2-h_1)h_4}{(h_1+h_2)h_2h_3(h_3+h_4)}f_2-\frac{H_2(H_2-h_1)(h_3+h_4)}{H_1(h_2+h_3)h_3h_4}f_3$$

$$+\frac{(2H_2-h_1)(h_3+h_4)h_4+H_2(H_2-h_1)(h_3+2h_4)}{H_2(H_2-h_1)(h_3+h_4)h_4}f_4, \qquad (25)$$

where $H_2 = h_1 + h_2 + h_3 + h_4$. Also the truncation errors corresponding to (23)-(25) are as given below:

$$E_4'(x_0) = \frac{h_1(h_1+h_2)H_1H_2}{120} f^v(\xi) \tag{26}$$

$$E_4'(x_2) = \frac{(h_1+h_2)h_2h_3(h_3+h_4)}{120} f^v(\xi) \tag{27}$$

and

$$E_4'(x_4) = \frac{H_2(H_2-h_1)(h_2+h_3)h_4}{120} f^v(\xi) \tag{28}$$

The second and third order derivatives of the function $f(x)$ at $x = x_0,\ x_2$ and x_4 are

$$f''(x_0) = \frac{(3h_1+2h_2)(2H_2-h_4)+2H_1H_2+h_1(4H_2-2h_3-3h_4)}{h_1(h_1+h_2)(h_1+h_2+h_3)H_2} f_0 - \frac{2(h_1+h_2)(2H_2-h_4)+2H_1H_2}{h_1h_2(h_1+h_3)(h_2+h_3+h_4)} f_1 + \frac{2h_1(2H_2-h_4)+2H_1H_2}{(h_1+h_2)h_2h_3(h_3+h_4)} f_2 - \frac{2h_1(h_1+h_2+H_2)+2(h_1+h_2)H_2}{(h_1+h_2+h_3)(h_2+h_3)h_3h_4} f_3 + \frac{2h_1(2H_1-h_3)+2(h_1+h_2)H_1}{H_2(h_2+h_3+h_4)(h_3+h_4)h_4} f_4, \tag{29}$$

$$f''(x_2) = \frac{-2h_2(2h_3+h_4)+2h_3(h_3+h_4)}{h_1(h_1+h_2)(h_1+h_2+h_3)H_2} f_0 + \frac{2(h_1+h_2)(2h_3+h_4)-2h_3(h_3+h_4)}{h_1h_2(h_1+h_3)(h_2+h_3+h_4)} f_1 + \frac{2h_2(h_1+h_2)-2(h_1+2h_2)(2h_3+h_4)+2h_3(h_3+h_4)}{(h_1+h_2)h_2h_3(h_3+h_4)} f_2 + \frac{2(h_1+2h_2)(h_3+h_4)-2h_2(h_1+h_2)}{(h_1+h_2+h_3)(h_2+h_3)h_3h_4} f_3 + \frac{2(h_1+h_2)h_2-2(h_1+2h_2)h_3}{H_2(h_2+h_3+h_4)(h_3+h_4)h_4} f_4 \tag{30}$$

$$f''(x_4) = \frac{(h_2+3h_3+3h_4)h_4+(H_2-h_1)(2h_3+3h_4)}{h_1(h_1+h_2)(h_1+h_2+h_3)H_2} f_0$$

$$-\frac{(H_2+h_3+h_4)h_4+H_2(2h_3+3h_4)+(h_3+h_4)h_4}{h_1h_2(h_1+h_3)(h_2+h_3+h_4)}f_1$$

$$+\frac{2(2H_2-h_1)h_4+2(H_2-h_1)H_2}{(h_1+h_2)h_2h_3(h_3+h_4)}f_2-\frac{2(2H_2-h_1)(h_3+h_4)+2H_2(H_2-h_1)}{(h_1+h_2+h_3)(h_2+h_3)h_3h_4}f_3$$

$$+\frac{2(2H_2-h_1)(h_3+2h_4)+2(h_3+h_4)h_4+2H_2(H_2-h_1)}{H_2(h_2+h_3+h_4)(h_3+h_4)h_4}f_4, \quad (31)$$

$$f'''(x_0)=-\frac{6(4h_1+3h_2+2h_3+h_4)}{h_1(h_1+h_2)H_1H_2}f_0+\frac{6(3h_1+3h_2+2h_3+h_4)}{h_1h_2(h_1+h_3)(H_2-h_1)}f_1$$

$$-\frac{6(3h_1+2h_2+2h_3+h_4)}{(h_1+h_2)h_2h_3(h_3+h_4)}f_2+\frac{6(3h_1+2h_2+h_3+h_4)}{H_1(h_2+h_3)h_3h_4}f_3$$

$$-\frac{6(3h_1+2h_2+h_3)}{H_2(H_2-h_1)(h_3+h_4)h_4}f_4, \quad (32)$$

$$f'''(x_2)=\frac{6(h_2-2h_3-h_4)}{h_1(h_1+h_2)H_1H_2}f_0-\frac{6(h_1+h_2-2h_3-h_4)}{h_1h_2(h_1+h_3)(H_2-h_1)}f_1$$

$$+\frac{6(h_1+2h_2-2h_3-h_4)}{(h_1+h_2)h_2h_3(h_3+h_4)}f_2-\frac{6(h_1+2h_2-h_3-h_4)}{H_1(h_2+h_3)h_3h_4}f_3$$

$$+\frac{6(h_1+2h_2-h_3)}{H_2(H_2-h_1)(h_3+h_4)h_4}f_4, \quad (33)$$

$$f'''(x_4)=\frac{6(h_2+2h_3+3h_4)}{h_1(h_1+h_2)H_1H_2}f_0-\frac{6(h_1+h_2+2h_3+3h_4)}{h_1h_2(h_1+h_3)(H_2-h_1)}f_1$$

$$+\frac{6(h_1+2h_2+2h_3+3h_4)}{(h_1+h_2)h_2h_3(h_3+h_4)}f_2-\frac{6(h_1+2h_2+3h_3+3h_4)}{H_1(h_2+h_3)h_3h_4}f_3$$

$$+\frac{6(h_1+2h_2+3h_3+4h_4)}{H_2(H_2-h_1)(h_3+h_4)h_4}f_4. \quad (34)$$

Equations (29)-(34) represent the forward, central and backward difference formulae which are widely used to approximate the derivatives in practice. Also the fourth order derivative is given by

$$f^{iv}(x) = \frac{16}{h_1(h_1+h_2)H_1H_2} f_0 - \frac{16}{h_1h_2(h_1+h_3)(H_2-h_1)} f_1$$

$$+ \frac{16}{(h_1+h_2)h_2h_3(h_3+h_4)} f_2 - \frac{16}{H_1(h_2+h_3)h_3h_4} f_3$$

$$+ \frac{16}{H_2(H_2-h_1)(h_3+h_4)h_4} f_4 . \quad (35)$$

The truncation errors corresponding to (29)-(35) can be obtained by differentiating (4), as obtained in (26)-(28) corresponding to (23)-(25). As particular cases, if we put $h = h_1 = h_2 = h_3 = h_4$ in the above expressions (29)-(35), we obtained the results for equal sub-intervals as given below

$$f''(x_0) = \frac{35f_0 - 104f_1 + 114f_2 - 56f_3 + 11f_4}{12h^2} \quad (36a)$$

$$f''(x_2) = \frac{-f_0 + 16f_1 - 30f_2 + 16f_3 - f_4}{12h^2} \quad (36b)$$

$$f''(x_4) = \frac{11f_0 - 56f_1 + 114f_2 - 104f_3 + 35f_4}{12h^2} \quad (36c)$$

$$f'''(x_0) = \frac{-5f_0 + 18f_1 - 24f_2 + 14f_3 - 3f_4}{2h^3} \quad (36d)$$

$$f'''(x_2) = \frac{-f_0 + 2f_1 - 2f_3 + f_4}{2h^3} \quad (36e)$$

$$f'''(x_4) = \frac{f_0 - 14f_1 + 24f_2 - 18f_3 + 5f_4}{2h^3} \quad (36f)$$

and

$$f^{iv}(x_2) = \frac{f_0 - 4f_1 + 6f_2 - 4f_3 + f_4}{h^4}, \quad (36g)$$

which are exactly same as obtained by Singh and Thorpe [9]. Formulation of the above finite difference approximations clearly suggests that the finite difference formulae in terms of any number of points can be obtained with the help of equations (1) and (4). The mathematical expressions resulting due to further differentiation of equation (1) will give finite difference formulae for higher derivatives.

3. Conclusion

Here we have presented 3, 4, and 5 points explicit finite difference formulae along with the error terms, for approximating the first and higher order derivatives for unequally spaced data. By differentiating the equation (1), finite difference formulae for higher order derivatives can also be obtained. These formulae can be used directly to solve the ordinary and partial differential equations and will serve to approximate the derivatives at the unequally spaced grid points. A scientific program of the above formulae will facilitate the use of any finite difference formulae of desired derivatives.

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References

[1] F. B. Hildebrand, Introduction to Numerical Analysis, Mc Graw Hill, Inc, New York, 1974.

[2] Jianping Li, General explicit difference formulas for numerical differentiation, J. Comput. Appl. Math., 183 (2005), 29–52.

[3] I. R. Khan and R. Ohba, Closed-form expressions for the finite difference approximations of first and higher derivatives based on Taylor series, J. Comput. Appl.Math., 107 (1999a), 179–193.

[4] I. R. Khan and R. Ohba, Digital differentiators based on Taylor series, IEICE Trans. Fund. E82-A, 12 (1999b), 2822–2824.

[5] I. R. Khan and R. Ohba, New finite difference formulas for numerical differentiation, J. Comput. Appl. Math.,126 (2000), 269–276.

[6] I. R. Khan and R. Ohba, Mathematical proof of explicit formulas for tap-coefficients of Taylor series based FIR digital differentiators, IEICE Trans.Fund.E84-A, (6) (2001), 1581–1584.

[7] I. R. Khan and R. Ohba, Taylor series based finite difference approximations of higher-degree derivatives, J. Comput. Appl. Math., 154 (2003a), 115–124.

[8] I. R. Khan, R. Ohba and N. Hozumi, Mathematical proof of closed form expressions for finite difference approximations based on Taylor series, J. Comput. Appl. Math., 150 (2003b), 303-309.

[9] A. K. Singh and G. R. Thorpe, Finite difference formulae from Lagrange's interpolation formula, J. Scientific Research, 52 (2008), 263-270.

[10] A. K. Singh and G. R. Thorpe, Simpson's 1/3-rule of integration for unequal divisions of integration domain, J. Concrete Applicable Maths., 1(3) (2003), 247-252.

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