Conditional Expectation of Certain Distributions
of Record Values

A. I. Shawky¹* and R. A. Bakoban²

1 King Abdulaziz University, Girls College of Education, P.O. Box 32691, Jeddah 21438, Saudi Arabia
*Permanent address: Fac. of Eng. at Shoubra P.O. Box 1206, El Maadi 11728, Cairo, Egypt
aishawky@yahoo.com

2 King Abdulaziz University, Girls College of Education, Scientific Section, Department of mathematics, P.O. Box 4269, Jeddah 21491, Saudi Arabia
rbakoban@hotmail.com

Abstract

General classes of continuous distribution are characterized by considering the conditional expectation of function of record statistics. The specific distribution considered as a particular case of the general class of distribution are Weibull, Pareto, exponential (Exp), power function, Burr, beta of the first kind, rectangular, Rayleigh, Lomax, inverse Weibull (IW), exponentiated gamma (EG), exponentiated Weibull (EW), exponentiated Pareto (EP), generalized Rayleigh (GR) and exponentiated exponential (EE).

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1. Introduction

The record values have been in process of development for many years and has become increasingly important. Articles relating to this area have appeared in numerous different publications. Many authors have studied record values; for example, Ahsanullah (1995) and Arnold et al. (1998). Some inferences based on record values were made for specific distributions by Balakrishnan and Chan (1994),

In this paper a general classes of distributions have been characterized through conditional expectation of lower record values.

Let \( X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)} \) be the first \( n \) lower record values from a population whose p.d.f. \( f(x) \) and c.d.f. \( F(x) \). Let \( H(x) = -\log F(x), h(x) = \frac{f(x)}{F(x)} \). Then, the p.d.f. of the \( X_{L(m)}, m=1,2,\ldots \), is given by (see Ahsanullah (1995) and Arnold et al. (1998))

\[
f_m(x) = \frac{H^{m-1}(x) f(x)}{\Gamma(m)}, \quad -\infty < x < \infty, \quad (1.1)
\]

where \( \Gamma(.) \) is a gamma function,

and the joint p.d.f. of two lower records \( X_{L(m)} \) and \( X_{L(n)}, m < n, m, n=1,2,\ldots \) is given by (see Ahsanullah (1995) and Arnold et al. (1998))

\[
f_{m,n}(x,y) = \frac{1}{\Gamma(m)\Gamma(n-m)} H^{m-1}(x) [H(y) - H(x)]^{n-m-1} h(x) f(y), \quad -\infty < y < x < \infty, \quad (1.2)
\]

where

\[
h(x) = -\frac{dH(x)}{dx}.
\]

Let

\[
\mu_{m+1|m} = E[\phi(X_{L(m+1)})|X_{L(m)} = x] = \frac{1}{F(x)} \int_{-\infty}^{y} \phi(y) f(y) dy, \quad (1.3)
\]

and

\[
\mu_{m|m+1} = E[\phi(X_{L(m)})|X_{L(m+1)} = y] = \frac{m}{H^n(y)} \int_{\phi(x)H^{m-1}(x)h(x)dx}, \quad (1.4)
\]

where \( \phi(.) \) is a monotonic, continuous and differentiable function.

2. Main results

Theorem 1

Let \( X \) be an absolutely continuous random variables with c.d.f. \( F(x) \) and p.d.f. \( f(x) \). Suppose \( F(x) < 1 \) for \( x \in (\alpha, \beta), F(\alpha) = 0 \) and \( F(\beta) = 1 \). Then (Table 1)
Conditional expectation of certain distributions

\[ F(x) = [a\phi(x) + b]^c \]  \hspace{1cm} (2.1)

If and only if

\[ \mu_{m+1} = \frac{1}{c + 1} \left[ c\phi(x) - \frac{b}{a} \right] \]  \hspace{1cm} (2.2)

where \( \phi(.) \) is a monotonic, continuous and differentiable function on \((\alpha, \beta)\) and \(a \neq 0, c > 0, b \) are finite constants.

**Proof**

It is clear that

\[ \mu_{m+1} = \frac{1}{F(x)} \int_{\alpha}^{x} \phi(y) dF(y). \]  \hspace{1cm} (2.3)

Integrating (2.3) by parts by treating \( \phi(y) \) the differential part we get

\[ \mu_{m+1} = \phi(x) - \frac{1}{F(x)} \int_{\alpha}^{x} \phi(y) dF(y). \]  \hspace{1cm} (2.4)

Using (2.1), we obtain

\[ \mu_{m+1} = \phi(x) - \frac{1}{F(x)} \int_{\alpha}^{x} [a\phi(y) + b]^c \phi'(y) dy \]

\[ = \phi(x) - \frac{1}{a(c + 1)F(x)} \{ [a\phi(x) + b]^{c+1} - [a\phi(\alpha) + b]^{c+1} \}. \]

Since \( F(\alpha) = 0 \), we get

\[ \mu_{m+1} = \phi(x) - \frac{1}{a(c + 1)} [a\phi(x) + b]. \]

After some simplification, we get (2.2). Then the necessary condition is proved. To prove the sufficient condition, from (2.2) and (1.3), we obtain

\[ \frac{1}{F(x)} \int_{\alpha}^{x} \phi(y) f(y) dy = \frac{1}{c + 1} \left[ c\phi(x) - \frac{b}{a} \right]. \]

Taking the derivative with respect to \( x \) we get

\( (c + 1)\phi(x)f(x) = f(x)[c\phi(x) - b / a] + cF(x)\phi'(x) \)

which gives

\[ \frac{a\phi(x) + b}{a} f(x) = cF(x)\phi'(x) \]

and then

\[ \frac{f(x)}{F(x)} = \frac{ac\phi'(x)}{a\phi(x) + b}. \]

Thus, the theorem is proved.

**Theorem 2**

Let \( X \) be an absolutely continuous random variables with c.d.f. \( F(x) \) and p.d.f. \( f(x) \). Suppose \( F(x) < 1 \) for \( x \in (\alpha, \beta) \), \( F(\alpha) = 0 \) and \( F(\beta) = 1 \). Then

\[ F(x) = [a\phi(x) + b]^c \]
if and only if
\[
\mu_{m+1|n+2} = \frac{(m+1)c}{H(y)}[\mu_{m|n+1} - \phi(y)] - \frac{b}{a}, \quad m \geq 1
\]  
(2.5)

and
\[
\mu_{m}^\beta = \frac{c}{H(y)}[\phi(\beta) - \phi(y)] - \frac{b}{a},
\]  
(2.6)

where \( \phi(.) \) is a monotonic, continuous and differentiable function on \((\alpha, \beta)\) and \(a \neq 0, c > 0, b \) are finite constants.

**Proof**

It is clear that
\[
\mu_{m|n+1} = \frac{m}{H^m(y)} \int_\beta^\phi(x)H^{m-1}(x)h(x)dx.
\]  
(2.7)

Integrating (2.7) by parts by treating \( \phi(x) \) the differential part we get
\[
\mu_{m|n+1} = \phi(y) + \frac{1}{H^m(y)} \int_\beta^\phi H^m(x)\phi'(x)dx.
\]  
(2.8)

In view of (2.1), we have
\[
\phi'(x) = \frac{[a\phi(x) + b]f(x)}{acF(x)}.
\]  
(2.9)

Using (2.9) in (2.8), we obtain
\[
\mu_{m|n+1} = \phi(y) + \frac{1}{cH^m(y)} \int_\beta^\phi(x)H^m(x)h(x)dx
\]
\[
+ \frac{b}{acH^m(y)} \int_\beta^\phi H^m(x)h(x)dx.
\]

Since
\[
\mu_{m+1|n+2} = \frac{m+1}{H^{m+1}(y)} \int_\beta^\phi(x)H^m(x)h(x)dx,
\]
then we get
\[
\mu_{m|n+1} = \phi(y) + \frac{H(y)}{(m+1)c} \mu_{m+1|n+2} + \frac{bH(y)}{(m+1)ac}.
\]

After some simplification, we get (2.5). Then the necessary condition is proved. To prove the sufficient condition, from (2.5) and (1.4), we obtain
\[
\frac{m}{H^m(y)} \int_\beta^\phi(x)H^{m-1}(x)h(x)dx = \phi(y) + \frac{bH(y)}{(m+1)ac}
\]
\[
+ \frac{1}{cH^m(y)} \int_\beta^\phi(x)H^m(x)h(x)dx.
\]

Taking the derivative with respect to \( y \) we get
Conditional expectation of certain distributions

\[
\frac{f(y)}{F(y)} = \frac{ac\phi'(y)}{a\phi(y) + b}.
\]

Thus, the theorem is proved.

For the necessary part in (2.6). It is clear that

\[
\mu_{|\beta} = \frac{1}{H(y)} \int_{y}^{\beta} \phi(x)h(x)dx.
\]

But, from (2.9)

\[
h(x) = \frac{ac\phi'(x)}{a\phi(x) + b}.
\]

Then,

\[
\mu_{|\beta} = \frac{c}{H(y)} \int_{y}^{\beta} \phi'(x)dx - \frac{cb}{H(y)} \int_{y}^{\beta} \frac{\phi'(x)}{a\phi(x) + b} dx.
\]

After some simplification, we get (2.6). Then the necessary condition is proved. To prove the sufficient condition, from (2.6), we obtain

\[
\frac{1}{H(y)} \int_{y}^{\beta} \phi(x)h(x)dx = \frac{c}{H(y)} [\phi(\beta) - \phi(y)] - \frac{b}{a}.
\]

Taking the derivative with respect to \( y \) and after some simplification, we get (2.1). Thus, the theorem is proved.

By repeatedly appealing the recurrence relation (2.5), we derive the following relation in terms of \( \mu_{|\beta} \):

\[
\mu_{|\beta,m+1} = m! \left( \frac{c}{H(y)} \right)^{m-1} \mu_{|\beta} - \phi(y) \sum_{i=1}^{m-1} \left( \frac{c}{H(y)} \right)^{m-i} \prod_{j=i+1}^{m} j
\]

\[
- \left[ 1 + \sum_{i=1}^{m-2} \left( \frac{c}{H(y)} \right)^{m-i} \prod_{j=i+2}^{m} j \right] \frac{b}{a}.
\]
Table 1
Examples of (2.1) distributions

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>(\phi(x))</th>
</tr>
</thead>
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<tr>
<td>-1</td>
<td>1</td>
<td>(\theta)</td>
<td>(e^{-x}(x+1))</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>(\theta)</td>
<td>(e^{-\delta x})</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>(\lambda)</td>
<td>(e^{-\left(x/\delta\right)^p})</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>(\theta)</td>
<td>((1+x)^{-\delta})</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>(\theta^p)</td>
<td>(e^{-x^{-p}})</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>(\alpha)</td>
<td>(e^{-x^2})</td>
</tr>
<tr>
<td>((\delta - \omega)^{-1})</td>
<td>0</td>
<td>1</td>
<td>(x - \omega)</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>(\theta)</td>
<td>((1 + \delta x^p)^{-\omega})</td>
</tr>
<tr>
<td>(-\delta^p)</td>
<td>1</td>
<td>1</td>
<td>((\delta - x)^p)</td>
</tr>
<tr>
<td>(-\delta^p)</td>
<td>1</td>
<td>1</td>
<td>(x^{-p})</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>((1 + \delta x)^{-\omega})</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>(F(x))</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>([1 - e^{-x}(x+1)]^\theta, x \geq 0)</td>
<td>EG</td>
</tr>
<tr>
<td>([1 - e^{-\delta x}]^\theta, x \geq 0)</td>
<td>EE</td>
</tr>
<tr>
<td>([1 - e^{-\left(x/\delta\right)^p}]^\theta, x \geq 0)</td>
<td>EW</td>
</tr>
<tr>
<td>([1 - (1+x)^{-\delta^p}]^\theta, x \geq 0)</td>
<td>EP</td>
</tr>
<tr>
<td>(e^{-\left(\theta/(\delta x)\right)^p}, x \geq 0)</td>
<td>IW</td>
</tr>
<tr>
<td>([1 - e^{-x^2}]^\theta, x \geq 0)</td>
<td>Burr X</td>
</tr>
<tr>
<td>(\left(\frac{x-\omega}{\delta-\omega}\right)^\theta, \omega \leq x &lt; \delta)</td>
<td>Rectangular</td>
</tr>
<tr>
<td>([1 - e^{-\left[(\delta x)^2\right]^\theta}]^\theta, x \geq 0)</td>
<td>GR</td>
</tr>
<tr>
<td>(1 - (1 + \delta x^p)^{-\omega}, x \geq 0)</td>
<td>Burr XII</td>
</tr>
<tr>
<td>(1 - \left(\frac{\delta - x}{\delta - \omega}\right)^\theta, \omega \leq x \leq \delta)</td>
<td>Beta of first kind</td>
</tr>
<tr>
<td>(1 - \delta^p x^{-p}, x \geq \delta)</td>
<td>Pareto</td>
</tr>
<tr>
<td>(1 - (1 + \delta x)^{-\omega}, x \geq 0)</td>
<td>Lomax</td>
</tr>
</tbody>
</table>
Theorem 3

Let \( X \) be an absolutely continuous random variables with c.d.f. \( F(x) \) and p.d.f. \( f(x) \). Suppose \( F(x) < 1 \) for \( x \in (\alpha, \beta) \), \( F(\alpha) = 0 \) and \( F(\beta) = 1 \). Then (Table 2)
\[
F(x) = [a - be^{-c\phi(x)}]^{-1}
\]
(2.10)

if and only if
\[
\mu_{m+1|n} = \phi(x) - \frac{a}{\lambda bc} E_{m+1|n}[e^{c\phi(y)}|Y_{L(n)} = x] + \frac{1}{\lambda c},
\]
(2.11)

where \( \phi(.) \) is a monotonic, continuous and differentiable function on \( (\alpha, \beta) \) and \( b, c \neq 0, \lambda > 0, a \) are finite constants.

Proof

It is clear that
\[
\mu_{m+1|n} = \frac{1}{F(x)} \int_{x}^{y} \phi(y) dF(y).
\]
(2.12)

Integrating (2.12) by parts by treating \( \phi(y) \) the differential part we get
\[
\mu_{m+1|n} = \phi(x) - \frac{1}{F(x)} \int_{x}^{y} F(y) d\phi(y).
\]
(2.13)

In view of (2.10), we have
\[
\phi'(y) = \frac{f(y)}{\lambda c F(y)} \left[ a e^{c\phi(y)} - \frac{b}{b} e^{c\phi(y)} - 1 \right].
\]
(2.14)

Using (2.14) in (2.13), we obtain
\[
\mu_{m+1|n} = \phi(x) - \frac{a}{\lambda bc F(x)} \int_{x}^{y} e^{c\phi(y)} f(y) dy + \frac{1}{\lambda c F(x)} \int_{x}^{y} f(y) dy.
\]
(2.15)

Relation (2.11) can be derived very easily from (2.15). Then the necessary condition is proved. To prove the sufficient condition, from (2.11) and (1.3), we have
\[
\int_{x}^{y} \phi(y) f(y) dy = F(x) \phi(x) - \frac{a}{\lambda bc} \int_{x}^{y} e^{c\phi(y)} f(y) dy + \frac{F(x)}{\lambda c}.
\]

Taking the derivative with respect to \( x \) we get
\[
\phi'(x) F(x) = \frac{f(x)}{\lambda c} [(a/b)e^{c\phi(x)} - 1]
\]
which gives
\[
\frac{\lambda bc \phi'(x) e^{-c\phi(x)}}{a - be^{-c\phi(x)}} = \frac{f(x)}{F(x)}.
\]

Thus, the theorem is proved.

Theorem 4

Let \( X \) be an absolutely continuous random variables with c.d.f. \( F(x) \) and p.d.f. \( f(x) \). Suppose \( F(x) < 1 \) for \( x \in (\alpha, \beta) \), \( F(\alpha) = 0 \) and \( F(\beta) = 1 \). Then
\[
F(x) = [a - be^{-c\phi(x)}]^{-1}
\]
if and only if
\begin{equation}
\mu_{m|n+1} = \phi(y) + \frac{aH(y)}{(m+1)\lambda bc} E_{n+1|n+2}[e^{c\phi(x)} | X_{L(m+2)} = y] - \frac{H(y)}{(m+1)\lambda c},
\end{equation}

where \( \phi(.) \) is a monotonic, continuous and differentiable function on \( (\alpha, \beta) \) and \( b, c \neq 0, \lambda > 0, a \) are finite constants.

**Proof**

It is clear that

\begin{equation}
\mu_{m|n+1} = \frac{m}{H^m(y)} \int_{y}^{\beta} \phi(x) H^{m-1}(x) h(x) dx.
\end{equation}

Integrating (2.17) by parts by treating \( \phi(x) \) the differential part we get

\begin{equation}
\mu_{m|n+1} = \phi(y) + \frac{1}{H^m(y)} \int_{y}^{\beta} H^m(x) \phi'(x) dx.
\end{equation}

From (2.14), we have

\begin{align*}
\mu_{m|n+1} &= \phi(y) + \frac{1}{\lambda c H^m(y)} \int_{y}^{\beta} [(a/b)e^{c\phi(x)} - 1] H^m(x) h(x) dx \\
&= \phi(y) + \frac{a}{\lambda bc H^m(y)} \int_{y}^{\beta} e^{c\phi(x)} H^m(x) h(x) dx \\
&\quad - \frac{1}{\lambda c H^m(y)} \int_{y}^{\beta} H^m(x) h(x) dx.
\end{align*}

Thus we get (2.16). Then the necessary condition is proved. To prove the sufficient condition, from (2.16) and (1.4), we obtain

\begin{align*}
\frac{m}{H^m(y)} \int_{y}^{\beta} \phi(x) H^{m-1}(x) h(x) dx &= \phi(y) - \frac{H(y)}{(m+1)\lambda c} \\
&\quad + \frac{aH(y)}{(m+1)\lambda bc} E_{n+1|n+2}[e^{c\phi(x)} | X_{L(m+2)} = y].
\end{align*}

Multiplying both sides by \( H^m(y) \) then by taking the derivative with respect to \( y \) we get

\[ \phi'(y) = \frac{1}{\lambda c} \left[ \frac{a}{b} e^{c\phi(y)} - 1 \right] h(y) \]

which gives

\[ \frac{f(y)}{F(y)} = \frac{\lambda bc \phi'(y)e^{-c\phi(y)}}{a - be^{-c\phi(y)}}. \]

Thus, the theorem is proved.
Table 2: Examples of (2.10) distributions

<table>
<thead>
<tr>
<th>a</th>
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<th>c</th>
<th>λ</th>
<th>φ(x)</th>
<th>F(x)</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
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<td>x - ln(1 + x)</td>
<td>[1 - e^{-x}(x + 1)]^\theta, x \geq 0</td>
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</tr>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>x</td>
<td>[1 - e^{-x\theta}]^\delta, x \geq 0</td>
<td>EE</td>
</tr>
<tr>
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<td>1</td>
<td>\omega^{-\delta}</td>
<td>0</td>
<td>x^\delta</td>
<td>[1 - e^{-(x/\omega)^\delta}]^\theta, x \geq 0</td>
<td>EW</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>\ln(1 + x)</td>
<td>[1 - (1 + x)^{-\delta}]^\theta, x \geq 0</td>
<td>EP</td>
<td></td>
</tr>
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<td>x^2</td>
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<td>x^\delta</td>
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</tr>
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<td>1</td>
<td>1</td>
<td>\alpha</td>
<td>x^\alpha</td>
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</tr>
<tr>
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<td>1</td>
<td>\omega</td>
<td>1</td>
<td>\ln(1 + \delta x^\rho)</td>
<td>1 - (1 + \delta x^\rho)^{-\omega}, x \geq 0</td>
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</tr>
<tr>
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<td>1</td>
<td>\delta</td>
<td>1</td>
<td>x^2</td>
<td>1 - e^{-\delta x^2}, x \geq 0</td>
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References


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