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Restarted Adomian's Decomposition Method for Duffing's Equation

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Abstract

In this paper we apply restarted Adomian decomposition method to solve Duffing's equation. Illustrative examples have been presented, to demonstrate the method. The results are compared with those of standard Adpmian decomposition method. The numerical results show that restarted method gives more accurate results all over the solution interval.

Keywords: Adomian decomposition method; Restarted Adomian method; Duffing equation

1 Introduction

The Adomian's decomposition method (ADM) is a solution method with a wide range of applications including the solution of algebraic, differential, integral and integro-differential equations or system of equations. This method was first introduced by Adomian [1, 2] in the beginning of the 1980's. In this method the solution is considered as an rapidly converging, infinite series. The convergence of the method proved by Y. Cherrualt et al. [3, 4].

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Restarted Adomian decomposition method (RADM), introduced by E. Babolian et al. [5] for the solution of algebraic equations. This method also useed to solve the integral equations [5] and can be applied to the systems of linear and nonlinear integral and algebraic equations. This method is based on the modifying of first term in ADM repeatedly that increases the rate of convergence of the response polynomial.

Mathematical modeling of many frontier physical systems leads to nonlinear ordinary differential equations (NODE). One of the most common physical NODE's, governs many oscillative systems, is the Duffing's equation. In this paper we attempt to applying the RADM to solve the Duffing's equation. Through several illustrative examples, we will show that convergence rate of the series solution of Duffing's problem using RADM is more accelerated than ADM.

2 Description of ADM for Duffing's equation

The Duffing's equation describes by second order ordinary differential equation with the common form

$$x'' + px' + p_1 x + p_2 x^3 = f(t) \tag{1}$$

$$x(0) = \alpha, \quad x'(0) = \beta \tag{2}$$

where p, p_1, p_2, α and β are real constants. Denoting $\frac{d^2}{dt^2}$ by L, we have L^{-1} as a two-fold integration given by $L^{-1}(.) = \int_0^t \int_0^t (.)dtdt$. Using the operator L, the equation (1) becomes

$$Lx = f(t) - px' - p_1 x - p_2 x^3. \tag{3}$$

Applying the invers operator L^{-1} on both sides of (3), yields

$$x = x(0) + x'(0)x + L^{-1}[f(t)] - pL^{-1}[x'] - p_1 L^{-1}[x] - p_2 L^{-1}[x^3] \tag{4}$$

In order to use ADM, let

$$x = \sum_{n=0}^{\infty} x_n, \tag{5}$$

and

$$N(x) = x^3 = \sum_{n=0}^{\infty} A_n, \tag{6}$$

where A_n's are Adomian polynomials of $x_0, x_1, \ldots, x_n$ that is given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{\infty} \lambda^i x_i)]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{7}$$

Using (6) in (3) we obtain the recurrence relation

$$x_0 = x(0) + x'(0)t + L^{-1}f(t), \tag{8}$$

$$x_{n+1} = -L^{-1}\frac{d}{dt}x_n - p_1 L^{-1}x_n - p_2 L^{-1}A_n, \quad n = 0, 1, 2, \ldots \tag{9}$$

In practice all terms of the series (5) can not be determined and so we use the following truncated series as an an approximation of the solution

$$\phi_k(t) = \sum_{n=0}^{k-1} x_n(x) \tag{10}$$

$$\lim_{k\to\infty} \phi_k(t) = x(t) \tag{11}$$

for the convergence of the Adomian decomposition sheme ,we refer the reader to [1, 4, 6]

3 Description of the RADM

Various formulas exist for calculating Adomian polynomials. The theorem extended in [5] shows the dependence of Adomian polynomials and correspondingly ADM on x_0. In the new algorithm the x_0 will be updated in each step. This can be done by adding a proper function on both sides of equation (4). If g denotes this proper function, then

$$x + g = f + g + N \tag{12}$$

where

$$f = x(0) + x'(0)t + L^{-1}f(t),$$

$$N(x) = -pL^{-1}\frac{d}{d_n}x(t) - p_1 L^{-1}x(t) - p_2 L^{-1}[x^3]$$

To solve the problem (1),we use the modified Adomian method. Let x and $N(x)$ be decomposed as (5)and (6). Then we can obtaion x_i as

$$x_0 = g,$$

$$x_1 = f - g + A_0,$$

$$x_2 = A_1,$$

$$\vdots$$

$$x_n = A_{n-1},$$

and introduce the following algorithm named "restarted Adomian method".

3.1 The algorithm

choose small natural numbers m,n.

Step1: Apply the Adomian method to Equation (4) and calculate $x_0, x_1, \ldots x_n$.

Set $\phi^1 = x_0 + x_1 + \ldots + x_n$.

Step2:

$$For\ i = 2 : m,$$

$$g = \phi^{i-1},$$

$$x_0 = g,$$

$$x_1 = f - g + A_0,$$

$$x_2 = A_1,$$

$$\vdots$$

$$x_n = A_{n-1}. \tag{13}$$

$$set\phi^i = x_0 + x_1 + \ldots + x_n. \tag{14}$$

$$end\ of for. \tag{15}$$

ϕ^m can be considered as approximate solution of Equation (1).

Note.Usually we choose m small, say $2 \leq m \leq 5$, so we calculate A_n for small values of n.

3.2 Examples

Example 1: Consider the Duffing's equation

$$x'' + x' + x + x^3 = cos^3 t - sint \qquad x(0) = 1,\ x'(0) = 0, \tag{16}$$

with the exact solution $x(t) = cost$. Considering the Maclaurin series of the excitation term

$$cos^3 t - sint \simeq 1 - t - \frac{3t^2}{2} + \frac{t^3}{6} + \frac{7t^4}{8}. \tag{17}$$

and using (8) and (9), we have

$$x_0 = 1 + \int_0^t \int_0^t (1 - t - \frac{3t^2}{2} + \frac{t^3}{6} + \frac{7t^4}{8}) dt dt,$$

$$x_{i+1} = \int_0^t \int_0^t y_i dt dt - \int_0^t \int_0^t A_i dt dt - \int_0^t \int_0^t \frac{d}{dt}(x_i) dt dt$$

Calculating seven terms of response series, an approximate solution of the problem will be obtain as

$$x(t) \simeq \phi_7 = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots + 3.75303 \times 10^{-36} t^{90} \quad (18)$$

Applying the algorithm of RADM with $n = 2$ and $m = 3$ we obtain

Step1

$$\phi^1 = x_0 + x_1 + x_2 = 1 - \frac{t^2}{2} + \frac{t^4}{8} + \frac{t^5}{8} + \frac{91t^6}{720} + \ldots + 1.48518 \times 10^{-13} t^{34} \quad (19)$$

Step2

$$\phi^2 = \phi^1 + x_3 + x_4 = 1 - \frac{t^2}{2} + \frac{t^4}{24} + \frac{t^6}{720} + \frac{2t^7}{315} + \ldots + 6.72279 \times 10^{-74} t^{174} \quad (20)$$

Step3

$$\phi^3 = \phi^2 + x_5 + x_6 = 1 - \frac{t^2}{2} + \frac{t^4}{24} + \frac{t^6}{720} + \frac{2t^7}{315} + \frac{61t^8}{5760} + \ldots + 1.970194563 \times 10^{-372} t^{874} \quad (21)$$

Obviously ϕ^3 is more accurate than the approximate solution (17) obtained by ADM. Note that in ADM we used $A_0, A_1, \ldots, A_7$ but in the new method we obtained a better approximation by calculating only A_0, A_1,three tims.

In order to compare the ADM and RADM, we calculate the absolute errors of the methods relative to the exact solution as shown in table 1. In this table the AE and RE show the ADM and RADM errors, respectively.

Time	AE (seven iterations)	RE (m=3,n=2)
0	0	0
0.1	6.52592×10^{-11}	6.50542×10^{-11}
$o.2$	1.43091×10^{-8}	1.39908×10^{-8}
0.3	3.60643×10^{-7}	3.33461×10^{-7}
0.4	3.88776×10^{-6}	3.18738×10^{-6}
0.5	2.76892×10^{-5}	1.84034×10^{-5}
0.6	1.57552×10^{-4}	7.71567×10^{-5}
0.7	7.75436×10^{-4}	2.59193×10^{-4}
0.8	3.39250×10^{-3}	7.3952×10^{-4}
0.9	1.33183×10^{-2}	1.85895×10^{-3}
1	4.72370×10^{-2}	4.22155×10^{-3}

Table 1. The errors of ADM and RADM for the problem of example 1.

Example 2: As second example consider another version of Duffing's equation as

$$x'' + 2x' + x + 8x^3 = e^{-3t}, \qquad x(0) = \frac{1}{2},\ x'(0) = -\frac{1}{2}, \tag{22}$$

The exact solution of this equation is $x(t) = \frac{1}{2}e^{-t}$.

Considering the Maclaurin series of the excitation term

$$e^{-3t} \simeq 1 - 3t + \frac{9t^2}{2} - \frac{9t^3}{2} + \frac{27t^4}{8}, \tag{23}$$

and using the relations (8) and (9), we have

$$\begin{aligned} x_0 &= \frac{1}{2} - \frac{t}{2} + \int_0^t \int_0^t (1 - 3t + \frac{9t^2}{2} - \frac{9t^3}{2} + \frac{27t^4}{8})dtdt, \\ x_{i+1} &= -\int_0^t \int_0^t x_i dtdt - 8\int_0^t \int_0^t A_i dtdt - 2\int_0^t \int_0^t \frac{d}{dt}(x_i)dtdt, \end{aligned} \tag{24}$$

from which the approximate polynomial solution can be obtained up to desiredterm. We apply restarted and standard ADM with m=3 and n=2

for this example. The polynomials will not be listed for brivety. The errors of two methods are shown in table 2.

Time	AE (seven iterations)	RE (m=3,n=2)
0	0	0
0.1	4.5384×10^{-9}	4.53786×10^{-9}
$o.2$	5.47739×10^{-7}	5.47180×10^{-7}
0.3	8.85108×10^{-6}	8.81931×10^{-6}
0.4	6.29771×10^{-5}	6.24291×10^{-5}
0.5	2.86678×10^{-4}	2.81780×10^{-4}
0.6	9.86268×10^{-4}	9.57433×10^{-4}
0.7	2.80273×10^{-3}	2.67552×10^{-3}
0.8	6.93681×10^{-3}	6.48184×10^{-3}
0.9	1.54734×10^{-2}	1.40831×10^{-2}
1	3.18546×10^{-2}	2.80807×10^{-2}

Table 2. The errors of ADM and RADM for the problem of example 2.

4 Conclusion

In this paper, we apply the restarted and standard ADM to approximate the solutions of Duffing's equation. Considering the errors listed in the tables 1 and 2, shows that RADM tends to more accurate approximation of the equations. The better performance of the RADM relative to ADM can be seen all over the solution interval.

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