Neighborhoods of a Certain Family of New Subclass with Negative Coefficients

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Abstract

By making use of the familiar concept of neighborhoods of analytic and univalent functions, the authors prove coefficient bounds and distortion inequalities, and associated inclusion relations for the \((n, \delta)\)-neighborhoods of a family of multivalent functions with negative coefficients.

Mathematics Subject Classification: 30C45

Keywords: Analytic functions, Starlike functions, Convex functions, \((n, \delta)\)-neighborhoods, Inclusion relations, Distortion inequalities

1 Introduction and Motivation

Let \( \mathcal{A} \) be the class of functions \( f \) normalized by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \).

As usual, we denote by \( S \) the subclass of \( \mathcal{A} \), consisting of functions which are also univalent in \( D \). We recall here the definitions of the wellknown classes of starlike function and convex functions respectively:

\[
S^* = \left\{ f \in \mathcal{A} : \Re \left( \frac{zf'(z)}{f(z)} \right) > 0, z \in D \right\},
\]
\[ K = \left\{ f \in A : \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, z \in D \right\}. \]

Let \( w \) be a fixed point in \( D \) and \( A(w) = \{ f \in H(D) : f(w) = f'(w) - 1 = 0 \} \). In [18], Kanas and Ronning introduced the following classes

\[ S_w = \{ f \in A(w) : f \text{ is univalent in } D \} \]

\[ SV_w^* = \left\{ f \in A(w) : \text{Re} \left( \frac{(z-w)f'(z)}{f(z)} \right) > 0, z \in D \right\}, \quad (2) \]

\[ CV_w = \left\{ f \in A : 1 + \left( \text{Re} \left( \frac{(z-w)f''(z)}{f'(z)} \right) \right) > 0, z \in D \right\}. \quad (3) \]

Later Acu and Owa [1] studied the classes extensively.

The class \( S_w^* \) is defined by geometric property that the image of any circular arc centered at \( w \) is starlike with respect to \( f(w) \) and the corresponding class \( S_w^c \) is defined by the property that the image of any circular arc centered at \( w \) is convex. We observed that the definitions are somewhat similar to the ones introduced by Goodman in [15] and [16] for uniformly starlike and convex functions, except that in this case the point \( w \) is fixed.

Let \( \Sigma_w \) denoted the subclass of \( A(w) \) consisting of the function of the form

\[ f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n(z-w)^n, \quad (a_n \geq 0, \ z \neq w). \quad (4) \]

The function \( f \) in \( \Sigma_w \) is said to be starlike function of order \( \beta \) if and only if

\[ \text{Re} \left\{ -\frac{(z-w)f'(z)}{f(z)} \right\} > \beta \quad (z \in D) \quad (5) \]

for some \( \beta(0 \leq \beta < 1) \). We denote by \( S_w^*(\beta) \) the class of all starlike functions of order \( \beta \). Similarly, a function \( f \) in \( \Sigma_w \) is said to be convex of order \( \beta \) if and only if

\[ \text{Re} \left( -1 - \frac{(z-w)f''(z)}{f'(z)} \right) > \beta \quad (z \in D) \quad (6) \]

for some \( \beta(0 \leq \beta < 1) \). We denote by \( C_w(\beta) \) the class of all convex functions of order \( \beta \).
We note that the class \( S^*_0(\beta) \) and various other subclasses of \( S^*_w(\beta) \) have been studied rather extensively by Nehari and Netanyahu [23], Acu and Owa [1], Clunie [8], Pommerenke [25, 24], Miller [21], Royster [26], and others (cf., e.g., Bajpai [6], Goel and Sohi [14], Mogra et al. [22], Uralegaddi and Ganigi [30], Cho et al. [7], Aouf [5], and Uralegaddi and Somanatha ([32], [31]); see also Duren ([9], pp. 29 and 137), and Srivastava and Owa ([28], pp. 86 and 429).

For the function \( f \) in the class \( S_w \), we define

\[
I^0 f(z) = f(z),
\]

\[
I^1 f(z) = (z - w)f'(z) + \frac{2}{z - w},
\]

\[
I^2 f(z) = (z - w) \left(I^1 f(z)\right)' + \frac{2}{z - w},
\]

and for \( k = 1, 2, 3, \ldots \) we can write

\[
I^k f(z) = (z - w) \left(I^{k-1} f(z)\right)' + \frac{2}{z - w}
\]

\[
= \frac{1}{z - w} + \sum_{n=1}^{\infty} n^k a_n (z - w)^n. \quad (7)
\]

The differential operator \( I^k \) studied extensively by Ghanim and Darus [11, 12, 13] and in the case \( w = 0 \) was given by Frasin and Darus [10].

Further, Let \( T^*_w \) denoted the subclass of \( S_w \) consisting functions of the form

\[
f(z) = \frac{1}{z - w} - \sum_{n=1}^{\infty} a_n (z - w)^n, \quad (a_n \geq 0). \quad (8)
\]

Now, we define the \((n, \delta)\)- neighborhood of the function \( f \in T^*_w \) by

\[
N_{n,\delta} (f; g)
\]

\[
= \left\{ g \in T^*_w : g(z) = \frac{1}{z - w} - \sum_{n=1}^{\infty} b_n (z - w)^n \text{ and } \sum_{n=1}^{\infty} n |a_n - b_n| \leq \delta \right\}. \quad (9)
\]
In particular, for the identity function
\[ e(z) = \frac{1}{z-w} \] (10)
we immediately have
\[ N_{n,\delta}(e;g) = \left\{ g \in T^*_w : g(z) = \frac{1}{z-w} - \sum_{n=1}^{\infty} b_n (z-w)^n \quad \text{and} \quad \sum_{n=1}^{\infty} n |b_n| \leq \delta \right\} . \] (11)

There are many other work that have been done in neighborhood such as by Goodman [17], Ruscheweyh [27] and also see ([2], [3], [4], and [29]). The above concept of \((n, \delta)\)-neighborhoods was extended and applied recently to families of meromorphically multivalent functions by Liu and Srivastava ([19] and [20]).

Next, for the functions \(f_j\) \((j = 1, 2)\) given by
\[ f_j(z) = \frac{1}{z-w} - \sum_{n=1}^{\infty} a_{n,j} (z-w)^n . \] (12)

Let \(f_1 * f_2\) denote the Hadamard product (or convolution) of \(f_1\) and \(f_2\), defined by
\[ (f_1 * f_2)(z) = \frac{1}{z-w} - \sum_{n=1}^{\infty} a_{n,1} a_{n,2} (z-w)^n = (f_2 * f_1)(z) . \] (13)

An interesting unification of the classes \(S^*_w(\beta)\) and \(CV_w(\beta)\) is provided by the class \(S^*_w(\beta, \lambda, k)\) of functions \(f \in T^*_w\), which also satisfy the following inequality:
\[ \Re \left\{ \frac{I^{k+1}f(z) + \lambda I^{k+1}f(z)}{\lambda I^{k}f(z) + (1-\lambda)f(z)} \right\} > \beta \] (14)
\(((z-w) \in D; 0 \leq \beta < 1; 0 \leq \lambda \leq 1)\).

The main object of this paper is to derive several coefficient bounds and distortion inequalities, and associated inclusion relations for the \((n, \delta)\)-neighborhood of functions in the subclass \(S^*_w(k, \beta, \lambda, \zeta)\) of the class \(T^*_w\), which consists of functions \(f \in T^*_w\) satisfying the following nonhomogeneous Cauchy-Euler differential equation:
\[ I^{2}f(z) + 2(\mu+1)I^{1}f(z) + \mu(\mu+1)f(z) = \left(\mu^2 + 3\mu + 3\right)g(z) , \] (15)
\((f(z) \in S^*_w; g(z) \in S^*_w(\beta, \lambda, k) ; \mu > -1 (\mu \in R))\).


2 Coefficient Bounds And Distortion Inequalities

Theorem 2.1 Let the function $f \in T^*_w$ be defined by (8). Then the function $f$ is in the class $S^*_w(\beta, \lambda, k)$ if and only if

$$\sum_{n=1}^{\infty} \left(n^k (n\lambda - \beta \lambda + 1) + \beta (1 - \lambda)\right) a_n \leq 1 - \beta + \lambda$$  \hspace{1cm} (16)

($n \in \mathbb{N}; 0 \leq \beta < 1; 0 \leq \lambda \leq 1$).

The result is sharp with the external function given by

$$f(z) = \frac{1}{z - w} - \sum_{n=1}^{\infty} \frac{1 - \beta + \lambda}{n^k (n\lambda - \beta \lambda + 1) + \beta (1 - \lambda)} (z - w)^n.$$  \hspace{1cm} (17)

Proof: If $f \in S^*_w(\beta, \lambda, k)$, the condition (14) is satisfied. Then for $0 < |z - w| = r < 1$, equation (14) may be written as

$$\Re \left\{ \frac{I^k f(z) + \lambda I^{k+1} f(z)}{\lambda I^k f(z) + (1 - \lambda) f(z)} \right\} > \beta$$  \hspace{1cm} (18)

Now, we let

$$A(z) = I^k f(z) + \lambda I^{k+1} f(z)$$

and let

$$B(z) = \lambda I^k f(z) + (1 - \lambda) f(z)$$

Then (18) is equivalent to $|A(z) + (1 - \beta) B(z)| > |A(z) - (1 + \beta) B(z)|$ for $0 \leq \beta < 1$. For $A(z)$ and $B(z)$ as above, we have

$$|A(z) + (1 - \beta) B(z)| =$$

$$\left| \frac{2 + \lambda - \beta}{z - w} - \sum_{n=1}^{\infty} \left(n^k (\lambda n + \lambda + 1 - \beta) + (1 - \lambda - \beta + \beta \lambda)\right) a_n (z - w)^n \right|$$

$$\geq \frac{2 + \lambda - \beta}{|z - w|} - \sum_{n=1}^{\infty} \left(n^k (\lambda n + \lambda + 1 - \beta) + (1 - \lambda - \beta + \beta \lambda)\right) a_n |z - w|^n$$

and similarly

$$|A(z) - (1 + \beta) B(z)| <$$
\[
\frac{\beta - \lambda}{|z - w|} + \sum_{n=1}^{\infty} \left( n^k (\lambda n + \lambda - 1 + \lambda \beta) - (1 - \lambda + \beta - \beta \lambda) \right) a_n |z - w|^n.
\]

Therefore,

\[
|A(z) + (1 - \beta) B(z)| - |A(z) - (1 + \beta) B(z)| \geq \frac{2 + 2\lambda - 2\beta}{|z - w|} - \sum_{n=1}^{\infty} \left( 2n^k (\lambda n - \lambda \beta + 1) + 2\beta (1 - \lambda) \right) a_n |z - w|^n
\]

\[
\geq \frac{2(1 + \lambda - \beta)}{r} - \sum_{n=1}^{\infty} \left( 2n^k (\lambda n - \lambda \beta + 1) + 2\beta (1 - \lambda) \right) a_n r^n. \tag{19}
\]

letting \( r \to 1 \) in (19), we obtain

\[
\sum_{n=1}^{\infty} \left( n^k (n \lambda - \beta \lambda + 1) + \beta (1 - \lambda) \right) a_n \leq 1 - \beta + \lambda
\]

which yields (16).

On the other hand, we must have

\[
\Re \left\{ \frac{I^k f(z) + \lambda I^{k+1} f(z)}{\lambda I^k f(z) + (1 - \lambda) f(z)} \right\} > \beta,
\]

Upon choosing the values of \((z - w)\) on the positive real axis where \(0 < |z - w| = r < 1\), the above inequality reduce to

\[
\Re \left\{ \frac{1 - \beta + \lambda - \sum_{n=1}^{\infty} (n^k (n \lambda - \beta \lambda + 1) + \beta (1 - \lambda)) a_n r^{n+1}}{1 - \sum_{n=1}^{\infty} (n \lambda + 1 - \lambda) a_n r^{n+1}} \right\} > 0
\]

The above inequality reduce to

\[
\Re \left\{ 1 - \beta + \lambda - \sum_{n=1}^{\infty} (n^k (n \lambda - \beta \lambda + 1) + \beta (1 - \lambda)) a_n r^{n+1} \right\} > 0,
\]

Letting \( r \to 1 \), we get the desired result.

**Corollary 2.2** Let the function \( f \in T_w^* \) be defined by (8), and \( \lambda = 0 \). Then the function \( f \) is in the class \( S_w^* (\beta, \lambda, k) \) if and only if

\[
\sum_{n=1}^{\infty} (n^k + \beta) a_n \leq 1 - \beta. \tag{20}
\]

The result is sharp with the external function given by

\[
f(z) = \frac{1}{z - w} - \frac{1 - \beta}{n^k + \beta} (z - w)^n. \tag{21}
\]
Corollary 2.3 Let the function $f \in T_w^*$ be defined by (8), and $\lambda = 1$. Then the function $f$ is in the class $S_w^*(\beta, \lambda, k)$ if and only if
\[ \sum_{n=1}^{\infty} n^k (n - \beta + 1) a_n \leq 2 - \beta. \] (22)

The result is sharp with the external function given by
\[ f(z) = \frac{1}{z - w} - \frac{2 - \beta}{n^k (n - \beta + 1)} (z - w)^n. \] (23)

Theorem 2.4 Let the function $f$ given by (8) be in the class $S_w^*(\beta, \lambda, k)$. Then
\[ a_n \leq \frac{1 - \beta + \lambda}{n^k (n\lambda - \beta\lambda + 1) + \beta (1 - \lambda)}, \quad (a_n \geq 1) \] (24)
and
\[ na_n \leq \frac{1 - \beta + \lambda}{n^{k-1} (n\lambda - \beta\lambda + 1) + \beta (1 - \lambda)}, \quad (a_n \geq 1). \] (25)

Proof: By using Theorem 2.1, we find from (16) that
\[ n^k (n\lambda - \beta\lambda + 1) + \beta (1 - \lambda) a_n \leq \sum_{n=1}^{\infty} n^k (n\lambda - \beta\lambda + 1) + \beta (1 - \lambda) a_n \leq 1 - \beta + \lambda \]
which immediately yields the first assertion (24) of Theorem 2.4. By appealing to (16), we also have
\[ n^{k-1} (n\lambda - \beta\lambda + 1) + \beta (1 - \lambda) na_n \leq 1 - \beta + \lambda \]
which implies
\[ na_n \leq \frac{1 - \beta + \lambda}{n^{k-1} (n\lambda - \beta\lambda + 1) + \beta (1 - \lambda)}. \]
Eventually, we are led to the second assertion (25).

Our main distortion inequalities for functions in the class $S_w^*(\beta, \lambda, k, \mu)$ are given by Theorem 2.5.
Theorem 2.5 If \( f \in T_w^* \) is in the class \( S^*_w(\beta, \lambda, k, \mu) \), then

\[
|f(z)| \leq \frac{1}{|z-w|} + \frac{(\mu^2 + 3\mu + 3)(1 - \beta + \lambda)}{(\lambda - \lambda \beta + 1) + \beta (1 - \lambda) ((3 + 2\mu) + \mu (\mu + 1))} |z-w|
\]

and

\[
|f(z)| \geq \frac{1}{|z-w|} - \frac{(\mu^2 + 3\mu + 3)(1 - \beta + \lambda)}{(\lambda - \lambda \beta + 1) + \beta (1 - \lambda) ((3 + 2\mu) + \mu (\mu + 1))} |z-w|,
\]

\( ((z-w) \in D) \).

Proof: Suppose that \( f \in T_w^* \) is given by (8). Also let the function \( g \in S^*_w(\beta, \lambda, k) \), occurring in the nonhomogeneous Cauchy-Euler differential equation (15), be given as in the definitions (9) and (11) with, of course,

\[
b_n \geq 0, \quad (n = 1, 2, 3, \ldots).
\]

Then we readily find from (15) that

\[
a_n = \frac{(\mu^2 + 3\mu + 3)}{n (n + 2\mu + 2) + \mu (\mu + 1)} b_n, \quad (n = 1, 2, 3, \ldots).
\]

so that

\[
f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n = \frac{1}{z-w} + \sum_{n=1}^{\infty} \frac{(\mu^2 + 3\mu + 3)}{n (n + 2\mu + 2) + \mu (\mu + 1)} b_n (z-w)^n
\]

and

\[
|f(z)| \leq \frac{1}{|z-w|} + |z-w| \sum_{n=1}^{\infty} \frac{(\mu^2 + 3\mu + 3)}{n (n + 2\mu + 2) + \mu (\mu + 1)} b_n.
\]

Next, since \( g \in S^*_w(\beta, \lambda, k) \), the first assertion (24) of Theorem 2.4 yields the following coefficient inequality:

\[
b_n \leq \frac{1 - \beta + \lambda}{n^k (n\lambda - \beta\lambda + 1) + \beta (1 - \lambda)}, \quad (a_n \geq 1)
\]
which, in conjunction with (26), yields

\[ |f(z)| \leq \frac{1}{|z-w|} \]

\[ + \frac{(\mu^2 + 3\mu + 3) (1 - \beta + \lambda)}{((\lambda - \lambda\beta + 1) + \beta (1 - \lambda)) ((3 + 2\mu) + \mu (\mu + 1))} |z - w|. \tag{32} \]

the first assertion (26) of Theorem 2.5 follows at once from (32).

The second assertion (27) of Theorem 2.5 can be proven by similarly applying (29) and (31).

By setting \( \lambda = 0 \) and \( \lambda = 1 \) in Theorem 2.5, and using the relationships in (14), we arrive at Corollaries 2.6 and 2.7, respectively.

**Corollary 2.6** If the functions \( f \) and \( g \) satisfy the nonhomogeneous Cauchy-Euler differential (15) with \( g \in S_w^*(\beta) \), then

\[ \frac{1}{|z-w|} - \frac{(\mu^2 + 3\mu + 3) (1 - \beta)}{(1 + \beta) ((3 + 2\mu) + \mu (\mu + 1))} |z - w| \leq |f(z)| \]

\[ \leq \frac{1}{|z-w|} + \frac{(\mu^2 + 3\mu + 3) (1 - \beta)}{(1 + \beta) ((3 + 2\mu) + \mu (\mu + 1))} |z - w|, \tag{33} \]

\(((z-w) \in D)\).

**Corollary 2.7** If the functions \( f \) and \( g \) satisfy the nonhomogeneous Cauchy-Euler differential (15) with \( g \in CV_w^*(\beta) \), then

\[ \frac{1}{|z-w|} - \frac{(\mu^2 + 3\mu + 3)}{((3 + 2\mu) + \mu (\mu + 1))} |z - w| \leq |f(z)| \]

\[ \leq \frac{1}{|z-w|} + \frac{(\mu^2 + 3\mu + 3)}{((3 + 2\mu) + \mu (\mu + 1))} |z - w|, \tag{34} \]

\(((z-w) \in D)\).

### 3 Neighborhoods For The Classes \( S^*_w(\beta, \lambda, k) \)

and \( \psi(\beta, \lambda, k, \mu) \)

In this section, we determine inclusion relations for the classes \( S^*_w(\beta, \lambda, k) \) and \( S^*_w(\beta, \lambda, k; \mu) \) involving the \((n, \delta)\)-neighborhoods defined by (9) and (11).
Theorem 3.1 If \( f \in T^*_w \) is in the class \( S^*_w (\beta, \lambda, k) \), then
\[
S^*_w (\beta, \lambda, k) \subset N_{n, \delta} (e; f),
\]
where \( e(z) \) is given by (10) and
\[
\delta := \frac{1 - \beta + \lambda}{n^{k-1} (n \lambda - \beta \lambda + 1) + \beta (1 - \lambda)}.
\]

Proof: Assertion (35) would follow easily from the definition of \( N_{n, \delta} (e; f) \), which given by (11) with \( g \) replaced by \( f \), and the second assertion (25) of Theorem 2.4.

Theorem 3.2 If \( f \in T^*_w \) is in the class \( \psi (\beta, \lambda, k, \mu) \), then
\[
S^*_w (\beta, \lambda, k, \mu) \subset N_{n, \delta} (g; f),
\]
where \( g \) is given by (15) and
\[
\delta := \left( \frac{1 - \beta + \lambda}{n^k (\lambda n - \lambda \beta + 1) + \beta (1 - \lambda)} + \frac{(\mu^2 + 3 \mu + 3) (1 - \beta + \lambda)^2}{(n^{k+1} (\lambda n - \lambda \beta + 1) + \beta (1 - \lambda))^2 (n (n + 2 \mu + 2) + \mu (\mu + 1))} \right).
\]

Proof: Suppose that \( f \in \psi (\beta, \lambda, k, \mu) \). Then, upon substituting from (28) into the following coefficient inequality:
\[
\sum_{n=1}^{\infty} n |a_n - b_n| \leq \sum_{n=1}^{\infty} nb_n + \sum_{n=1}^{\infty} na_n \quad (a_n \geq 0; b_n \geq 0)
\]
we obtain
\[
\sum_{n=1}^{\infty} n |a_n - b_n| \leq \sum_{n=1}^{\infty} nb_n + \sum_{n=1}^{\infty} \frac{(\mu^2 + 3 \mu + 3) (1 - \beta + \lambda)}{(n^{k+1} (\lambda n - \lambda \beta + 1) + \beta (1 - \lambda))^2 (n (n + 2 \mu + 2) + \mu (\mu + 1))} nb_n.
\]
Next, since \( g \in S^*_w (\beta, \lambda, k) \), the second assertion (25) of Theorem 2.4 yields
\[
nb_n \leq \frac{1 - \beta + \lambda}{n^{k-1} (n \lambda - \beta \lambda + 1) + \beta (1 - \lambda)} \quad (n = 1, 2, 3, \ldots).
\]
Finally, by making use of (25) as well as (40) on the right-hand side of (39), we find that
\[
\sum_{n=1}^{\infty} n \left| a_n - b_n \right| \leq \sum_{n=1}^{\infty} \left( \frac{1 - \beta + \lambda}{n^k (\lambda n - \lambda \beta + 1) + \beta (1 - \lambda)} \right) \times \\
\left( 1 + \sum_{n=1}^{\infty} \frac{\left( \mu^2 + 3 \mu + 3 \right) (1 - \beta + \lambda)}{n^k (\lambda n - \lambda \beta + 1) + \beta (1 - \lambda)} \right) \\
\leq \left( \frac{1 - \beta + \lambda}{n^k (\lambda n - \lambda \beta + 1) + \beta (1 - \lambda)} \right)^+ \\
\left( \frac{\left( \mu^2 + 3 \mu + 3 \right) (1 - \beta + \lambda)^2}{n^{k+\frac{1}{2}} (\lambda n - \lambda \beta + 1) + \beta (1 - \lambda)} \right)^2 \left( n (n + 2 \mu + 2) + \mu (\mu + 1) \right) \right) := \delta
\]

ACKNOWLEDGEMENTS. The work presented here was supported by eScienceFund: 04-01-02-SF0425.

References


**Received: October, 2008**