Conical Compactness
in Locally Convex Spaces

A. Bourass and M. Oudadess

Département de Mathématiques
faculté des sciences, Rabat, Morocco
bourass.a@fsr.ac.ma
oudadessm@yahoo.fr

Abstract

A notion of conical compactness in locally convex spaces is introduced. It is weaker than that of compactness in general. It coincides with it in the finite dimensional case. After a rapid discussion of some properties of this notion, we use it to give characterizations of locally convex spaces with the Schur property, of Montel spaces and of finite dimensional ones.

Mathematics Subject Classification: 46A03, 46A25

Keywords: local convexity, boundedness, conical compactness, $c-r-$compactness, $c-r-$ compactness, Schur property, Montel space

I. Introduction

The notion of conical boundedness has been introduced for the first time by R.D. Bourgin ([1],1973) in the context of Banach spaces. In 1989, G. Isac [2] has extended it to Hausdorff locally convex spaces (l.c.s) . He also adapted a characterization of conically bounded subsets in Banach spaces to introduce, in l.c.s.’s., the notion of p-conical boundedness [2] which appears to be different from the one of conical boundedness. Another notion has been pointed out by G.Isac and M.Théra. Working on optimization problems, they introduce the conically compact subsets in l.c.s.’s [3] . A. Ould Bahya and M.Oudadess [4] studied different aspects of these notions and obtained, in particular, characterizations of some classical l.c.s’s. (locally bounded, Schur cones and spaces, Montel and semi-reflexive spaces).

To define a conically compact set $D$ Isac and Thera consider cones determined by compact subsets and ask for the boundedness of the intersection of these cones with $D$. We drop the compactness of the subsets putting, instead of it, only the boundedness but we ask for the relative compactness of the intersections (see preliminaries). This choice seems to be justified by a theorem of Riesz
type and a characterization of Montel spaces without any restriction, while the one obtained in [4] is valid only in the metrizable case. Moreover, our proofs are shorter which again indicates that the definition adopted may be the natural one.

II. Preliminaries and definitions In the sequel, $E$ will be a vector space endowed with a Hausdorff locally convex topology $\tau$ given by a family $(p_\lambda)_{\lambda \in \Lambda}$ of seminorms. The topological dual will be denoted $E'$. Put also
\[
\mathcal{C}_b(E) = \{ B \subset E : B \text{ is convex bounded and closed with } 0 \notin B \}
\]
\[
\mathcal{C}_k(E) = \{ B \subset E : B \text{ is convex compact with } 0 \notin B \}
\]
\[
K(B) = \{ tb : t \geq 0, b \in B \}, \text{ the cone of summet } 0 \text{ generated by } B.
\]
For a continuous seminorm $p_\lambda$ on $E$, an $f \in E'\setminus\{0\}$ and $\varepsilon > 0$ put
\[
\mathcal{C}(f, p_\lambda, \varepsilon) = \{ x \in E : p_\lambda(x) \leq 1, f(x) \geq \varepsilon \}
\]
and denote by $K(\mathcal{C}(f, p_\lambda, \varepsilon))$ the cone of summet 0 generated by $\mathcal{C}(f, p_\lambda, \varepsilon)$.

We recall the following notions.

**Definition 1** Let $D$ be a subset of $E$.
1) $D$ is said to be conically bounded (c-bounded) if $D \cap K(B)$ is bounded for every $B \in \mathcal{C}_b(E)$ (cf. [2]).
2) $D$ is said to be $p$-conically bounded (p.c-bounded) if there is a continuous seminorm $p_\lambda$ on $E$ such that $D \cap K(\mathcal{C}(f, p_\lambda, \varepsilon))$ is bounded for every $f \in E'\setminus\{0\}$ and $\varepsilon > 0$ (cf. [1]).
3) $D$ is said to be conically compact (c-compact) if $D \cap K(B)$ is compact for every $B \in \mathcal{C}_k(E)$ (cf. [3]).

Now here are the definitions we are adopting in this paper.

**Definition 2** Let $D$ be a subset of $E$.
1) $D$ is said to be conically relatively compact (c.r-compact) if $D \cap K(B)$ is relatively compact for every $B \in \mathcal{C}_b(E)$.
2) $D$ is said to be $p$-conically relatively compact (p.c.r-compact) if there is a continuous seminorm $p_\lambda$ on $E$ such that $D \cap K(\mathcal{C}(f, p_\lambda, \varepsilon))$ is relatively compact for every $f \in E'\setminus\{0\}$ and $\varepsilon > 0$.

**Remark 3** 1) One has $D \cap K(\mathcal{C}(f, p_\lambda, \varepsilon)) = \{ x \in D : \exists \alpha \in E, \exists \alpha > 0, x = \alpha a, p_\lambda(a) \leq 1, f(a) \geq \varepsilon \} = \{ x \in D : \exists \alpha > 0, p_\lambda(x) \leq \alpha, f(x) \geq \alpha \varepsilon \}$
   If $p_\lambda(x) > 0$ for every $x \in D$, one can take $\alpha = p_\lambda(x)$ and then $D \cap K(\mathcal{C}(f, p_\lambda, \varepsilon)) = \{ x \in D : f(x) \geq \varepsilon p_\lambda(x) \}$
2) Every relatively compact (hence every compact) subset is c.r-compact and p.c.r-compact. The converse is not true. Indeed, let $(E, \tau)$ be a l.c.s. admitting a sequence $(x_n)_n$ which tends to 0 for the weak topology but
not for $\tau$ e.g. $l^2(\mathbb{R})$. Considering a subsequence if need be, one assumes that there exists a sminorm $p_\lambda$ such that $p_\lambda(x_n) > \delta$ for every $n \geq 1$ for a given $\delta > 0$. With $y_n = [p_\lambda(x_n)]^{-1}x_n$, one has $y_n \to 0$ for the weak topology and $p_\lambda(y_n) = 1$. The subset $D = \{ny_n, n \geq 1\}$ is p.c.r-compact for the topology $\tau$ (see the proof of Proposition III.2. above) but it is not relatively compact; otherwise it should be bounded which is impossible because $p_\lambda(ny_n) = n$.


4) Every p.c.r-compact (resp. c.r-compact) subset is p.c.-bounded (resp. c-bounded). A bounded (hence a c-bounded) set is not necessarily c.r-compact. Take the unit ball of an infinite dimensional normed space.

5) In normed spaces, R.D. Bourgin [1] has established that c-boundedness is equivalent to p.c.-boundedness.

6) If the topologies $\tau_1$ and $\tau_2$ are compatible with the duality $(E, E')$ and if $\tau_1$ is finer than $\tau_2$, then c-compact and p.c.r-compact subsets for $\tau_1$ are also so for $\tau_2$.

7) In a general l.c.s. the closure of a p.c.r-compact subset is not necessarily p.c.r-compact. Indeed, consider the space $c_0$ of real sequences $(x_n)_n$ converging to zero endowed with the norm given by $\| (x_n)_n \| = \sup \{|x_n|, n \in \mathbb{N}\}$. Let $(e_n)_n$ be the canonical basis, where $e_n = (0, \ldots, 0, 1, 0, \ldots)$ with 1 in the $n^{th}$ rank and 0 elsewhere. Take the l.c.s. $(c_0, \sigma)$ where $\sigma = \sigma(c_0, l_1)$ is the weak topology. Notice that $(c_0, \| . \|)$ and $(c_0, \sigma)$ have the same c-bounded sets. Put $D = \{n(e_1 + e_{n,k}), n = 2, 3, \ldots \text{and } k = 2, 3, \ldots\}$ where $e_{n,k} = e_{\max(n,k)}$. Then $D$ is a p.c.r-compact subset the weak closure $\overline{D}_\sigma$ of which is not. Observe first that for any $f \in l_1 \setminus \{0\}$ and any $\varepsilon > 0$, the set $\{n : n^{-1}f(e_1 + ne_{n,k}) \geq \varepsilon; k = 2, 3, \ldots\}$ is finite. Also $\{n : n(e_1 + ne_{n,k}) \in K(C(f, \| . \|, \varepsilon)); k = 2, 3, \ldots\}$ is finite. Whence the first claim. For the second, observe that the sequence $n(e_1 + ne_{n,k})_{k \geq n}$ is weakly convergent to $ne_1$. So $\{ne_1 : n = 2, 3, \ldots\} \subset \overline{D}_\sigma \cap K(e_1)$. Hence $\overline{D}_\sigma$ is not c-bounded which is less than to be p.c.r-compact and even c.r-compact.

8) The convex hull of a p.c.r-compact subset is not necessarily so. Consider the same Banach space $c_0$ as in the previous remark and take $D = \{n(e_1 \pm ne_n), n = 1, 2, \ldots\}$. Bourgin proves in fact ([1]) that $D$ is a p.c.r-compact set the convex hull of which is not conically bounded. Hence the latter is not p.c.r-compact.

III. Main results
It is known that a p.c.-bounded set is c-bounded [4]. Here is the analogous statement in our context.

**Proposition 4**  
Every p.c.r-compact subset of a l.c.s. $E$ is c.r-compact. The converse is true in the normed case.

**Proof.** Let $D$ be a p.c.r-compact subset of $E$. There is a continuous seminorm $p_\lambda$ such that $D \cap K(\mathcal{C}(f,p_\lambda, \varepsilon))$ is relatively compact for every $f \in E' \setminus \{0\}$ and every $\varepsilon > 0$. We will show that for every $B \in \mathcal{C}_b(E)$, there is $g \in E' \setminus \{0\}$ and $\alpha > 0$ such that $K(B) \subset K(\mathcal{C}(g, p_\lambda, \alpha))$. By Hahn-Banach separation theorem, there is $g \in E' \setminus \{0\}$ and $\delta > 0$ such that $0 < \inf \{g(x), x \in B\}$. Put $\alpha = M_\lambda^{-1}\delta$ where $M_\lambda = \sup \{p_\lambda(x), x \in B\}$. One has $M_\lambda < +\infty$ since $B \in \mathcal{C}_b(E)$. For $x \in K(B)$, let $t > 0$ such that $tx \in B$. Then $p_\lambda(M_\lambda^{-1}tx) \leq 1$ and $g(M_\lambda^{-1}tx) \geq M_\lambda^{-1}\delta = \alpha$. Hence $M_\lambda^{-1}tx \in \mathcal{C}(g, p_\lambda, \alpha)$. Whence $x \in K(\mathcal{C}(g, p_\lambda, \alpha))$. For the converse, notice that the set $\mathcal{C}(f, \|\cdot\|, \varepsilon) \subset \mathcal{C}_b(E)$ when $(E, \|\cdot\|)$ is a normed space.

Recall that a l.c.s. is said to have the Schur property if every weakly convergent sequence to 0 is also convergent for the initial topology. In [1], R.D. Bourgin showed that a Banach space has the Schur property if and only if every c-bounded subset is bounded. A. Ould Bahya and M. Oudadess extended this result to l.c.s.'s replacing c-boundedness by p.c.-boundedness. We provide the analogous result in our setting.

**Proposition 5**  
Let $K$ be a convex cone of summit 0 in a l.c.s. $(E, \tau)$. The following are equivalent.

a) There is a p.c.r-compact subset of $K$ which is not bounded.

b) There is a sequence in $K$ which is convergent to 0 but not $\tau$-convergent.

**Proof.**  
a) $\Rightarrow$ b): Let $D$ be a subset of $K$ which is p.c.r-compact but not bounded. There is a continuous seminorm $p_\lambda$ on $E$ and a sequence $(x_n)_n \subset D$ such that $D \cap K(\mathcal{C}(f,p_\lambda, \varepsilon))$ is relatively compact for every $f \in E' \setminus \{0\}$ and every $\varepsilon > 0$ with $p_\lambda(x_n) > n$ for every $n$. By 1) of Remarks II.3, $\{x_n, n \in \mathbb{N}\} \cap K(\mathcal{C}(f,p_\lambda, \varepsilon)) = \{x_n, f(x_n) \geq \varepsilon p_\lambda(x_n)\}$. Being relatively compact, this set is finite for otherwise, it should not be bounded since $p_\lambda(x_n) > n$. Then putting $y_n = [p_\lambda(x_n)]^{-1}x_n$, the set $\{y_n, f(y_n) \geq \varepsilon\}$ is finite for each $f \in E' \setminus \{0\}$ and $\varepsilon > 0$. So $\limsup f(y_n) \leq 0$ for every $f \in E' \setminus \{0\}$. Whence the weak convergence of $(y_n)_n$ to 0. Now clearly $(y_n)_n$ does not converge for $\tau$ since $p_\lambda(y_n) = 1$, for every $n$.

b) $\Rightarrow$ a): Let $(x_n)_n$ be a sequence in $K$ which is weakly convergent to 0 but not convergent to 0 for the topology $\tau$. There is a $\delta > 0$ and a continuous seminorm $p_\lambda$ such that $p_\lambda(x_n) > \delta$ for every $n$. The
sequence \((y_n)_n\) with \(y_n = [p_\lambda(x_n)]^{-1} x_n\), converges also weakly to 0 for \(|f(y_n) - f(x_n)| \leq \delta^{-1} |f(x_n)|\). Moreover \(p_\lambda(y_n) = 1\), for every \(n\). The subset \(D = \{ny_n, n \in \mathbb{N}^*\}\) is in the cone \(K\) and one has \(D \cap K(C(f, p_\lambda, \varepsilon)) = \{ny_n, \exists \alpha_n > 0, n \leq \alpha_n \land f(y_n) \geq \varepsilon \frac{\alpha_n}{n}\} = \{ny_n, f(y_n) \geq \varepsilon\}\) for each \(f \in E'\setminus \{0\}\) and \(\varepsilon > 0\). But \(\{ny_n, f(y_n) \geq \varepsilon\}\) is finite for \(\lim f(y_n) = 0\). Hence \(D \cap K(C(f, p_\lambda, \varepsilon))\) is also finite and so compact. The subset \(D\) is not bounded since \(p_\lambda(ny_n) = n\).

**Corollary 6** Let \(K\) be a convex cone in a l.c.s. The following assertions are equivalent:

a) \(K\) has the Schur property.

b) Every p.c.r−compact subset of \(K\) is bounded.

In [4] it is shown that a metrizable l.c.s. is a Montel space, if and only if, every c−compact subset (in the sense of Definition II.1) is c−bounded. The authors deduce that a normed space is finite dimensional, if and only if, every c−compact subset is c−bounded (same definitions). With our definition, we obtain characterizations of not necessarily metrizable l.c.s’s which are of Montel, and also non a priori normed spaces which are finite dimensional. This is done using the class of p.c.r−compact subsets which is smaller than that of c.r−compact ones.

**Proposition 7** A l.c.s. is finite dimensional if and only if zero admits a p.c.r−compact neighborhood.

**Proof.** If \(E\) is finite dimensional, then 0 admits a compact neighborhood \(V\) which is of course p.c.r−compact. For the converse, let \(V\) a p.c.r−compact neighborhood of 0. One may suppose it open for any subset of a p.c.r−compact set is also p.c.r−compact. By hypothesis there is a continuous seminorm \(p_\lambda\) such that \(V \cap K(C(f, p_\lambda, \varepsilon))\) is relatively compact for every \(f \in E', f \neq 0\), and every \(\varepsilon > 0\). Let \(x_0 \in V\) such that \(p_\lambda(x_0) > 0\). Since \(V\) is open, it is a neighborhood of \(x_0\). Choose \(g \in E'\) and \(\varepsilon_0 > 0\) such that \(g\left(\left[2p_\lambda(x_0)\right]^{-1} x_0\right) > \varepsilon_0\). One has \(p_\lambda\left(\left[2p_\lambda(x_0)\right]^{-1} x_0\right) < 1\) and \(\left[2p_\lambda(x_0)\right]^{-1} x_0 \in \{x : p_\lambda(x) < 1 \land g(x) > \varepsilon_0\}\) \(\subset C(g, p_\lambda, \varepsilon_0)\). But \(\{x : p_\lambda(x) < 1 \land g(x) > \varepsilon_0\}\) is open, hence \(\left[2p_\lambda(x_0)\right]^{-1} x_0 \in int C(g, p_\lambda, \varepsilon_0)\) which is contained in \(int K(C(g, p_\lambda, \varepsilon_0))\). Whence \(x_0 \in int K(C(g, p_\lambda, \varepsilon_0))\) and we do have \(x_0 \in V \cap int K(C(g, p_\lambda, \varepsilon_0)) \subset V \cap K(C(g, p_\lambda, \varepsilon_0))\). So \(V \cap K(C(g, p_\lambda, \varepsilon_0))\) is a relatively compact neighborhood of \(x_0\). One concludes by Riesz theorem.

**Proposition 8** A l.c.s. \(E\) is a Montel space if and only if every p.c−bounded subset of \(E\) is p.c.r−compact.
Proof. Let $B$ be a bounded subset of $E$. By a translation if need be, we may suppose that $0 \notin \overline{B}$. By the Hahn-Banach theorem, there is $f \in E'$ and $\varepsilon > 0$ such that $0 < \varepsilon < f(x)$ for every $x \in B$. Let $p_\lambda$ be a continuous semi-norm such that $p_\lambda(x) \leq 1$ for every $x \in B$. Then $B \subset C(f,p_\lambda,\varepsilon)$ and so $B \cap K(C(f,p_\lambda,\varepsilon)) = B$. Being bounded, $B$ is $p.c.$-bounded and so it is $p.c.r.$-compact by hypothesis. Hence $B$ is relatively compact. The converse is obvious.

Actually, one can say more.

Proposition 9  
In a l.c.s. $E$, the following assertions are equivalent.

a) $E$ is a Montel space

b) Every $p.c.$-bounded subset of $E$ is $p.c.r.$-compact.

c) Every $c.$-bounded subset of $E$ is $c.r.$-compact.

Bibliography

Math., Vol.44, n°2 (1973), 411-419


post-critical equilibrium state of a thin elastic plate, J. Opt. Theory


Received: March 23, 2007