A New Generalization of
Bojanov Varma’s Inequality

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Abstract

We prove an inequality of the form

\[ \| x^{(k)} \|_2^2 \leq A \| x^{(r)} \|_2^2 + B \| x \|_2^2 \]  

(1)

for \( k, r \in \mathbb{N} \cup \{0\} \), \( 0 \leq k < r \). For a given \( A \geq 0 \) we find the infimum \( B \geq 0 \) such that (0.1) holds for all sufficiently smooth functions.

Keywords: Kolmogorov’s type inequality, Hilbert space, Hermite polynomials, Parsevale’s equality

1 Introduction

Let \( L_{2,e^{-t^2}} (R) \) be the space of all measurable functions \( x : R \rightarrow R \) such that

\[ \| x \|_2^2 = \| x \|_{2,e^{-t^2}}^2 := \int_{-\infty}^{\infty} |x(t)|^2 e^{-t^2} dt < \infty. \]

Let \( L_{2,e^{-t^2}}^r (R) \) be the space of all functions \( x : R \rightarrow R \) such that \( x^{(r)} \in L_{2,e^{-t^2}} (R) \), and finally let \( L_{2,e^{-t^2}}^r (R) = L_{2,e^{-t^2}} (R) \cap L_{2,e^{-t^2}}^r (R) \).

For algebraic polynomials of degree \( n \) (we will denote the set of such polynomials by \( \pi_n \) ) exact inequalities of the form (1) for \( r = 2, 3, 4 \) and \( 0 \leq k < r \leq n \) was investigated by Varma [6]. Later Bojanov and Varma [2] gave a family of exact inequalities of the form (1) for each \( r \leq n \), and \( 0 \leq k < r \).

Let \( H_n(t) \) be the Hermite polynomial of degree \( n \). Set \( \overline{T}_n(t) = \frac{H_n}{\| H_n \|} \). Taking into account the equality \( H_n'(t) = 2n H_{n-1}(t) \), we obtain that \( \| \overline{T}_n \|_2^2 = 2^n \psi_k(n) \), where \( \psi_k(t) = t(t-1)(t-2)...(t-k+1) \). We will give the complete solution of the problem on exact constants in the inequality (1).
2 Main Results

Theorem 1 Let $k, r \in N, k < r$. Then

\[ \|x^{(k)}\|^2 \leq A \|x^{(r)}\|^2 + \left( \left\| \mathcal{H}_{v_0}^{(k)} \right\|^2 - A \left\| \mathcal{H}_{v_0}^{(r)} \right\|^2 \right) \|x\|^2 \]

\[ = A \|x^{(r)}\|^2 + \psi(A, v_0) \|x\|^2 \]  

(2)

holds for any $0 \leq A \leq \frac{\left| \mathcal{H}_{v_0}^{(k)} \right|^2 - \left| \mathcal{H}_{v_0}^{(r)} \right|^2}{\left| \mathcal{H}_{v_0}^{(v)} \right|^2}$ and any $x \in L^r_{2, 2, e^{-r^2}}(R)$ with

$\varphi(v_0 + 1) \leq A \leq \varphi(v_0)$, where $\varphi(v) = \frac{k}{r} \left| \mathcal{H}_v^{(k)} \right|^2$. For a given $A$ the constant in (2) is the best possible.

Proof Using Parseval’s equality, for any $x \in L^r_{2, 2, e^{-r^2}}(R)$, and any $0 \leq A \leq \frac{\left| \mathcal{H}_{v_0}^{(k)} \right|^2 - \left| \mathcal{H}_{v_0}^{(r)} \right|^2}{\left| \mathcal{H}_{v_0}^{(v)} \right|^2}$

\[ \|x^{(k)}\|^2 = \sum_{v \geq k} c_v^2(x) \left\| \mathcal{H}_v^{(k)} \right\|^2 \]

\[ \leq A \|x^{(r)}\|^2 + \sup_{v \geq r} \left( \left\| \mathcal{H}_v^{(k)} \right\|^2 - A \left\| \mathcal{H}_v^{(r)} \right\|^2 \right) \sum_{v \geq k} c_v^2(x) \]

\[ = A \|x^{(r)}\|^2 + \sup_{v \geq r} \psi(A, v) \|x\|^2 . \]

We aim to find the exact value of $\sup_{v \geq r} \psi(A, v)$. Consider the difference

$\delta_v = \psi(A, v) - \psi(A, v - 1) = 2^r (\psi_r(v) - \psi_r(v - 1)) (\varphi(v) - A)$

where $\varphi(v) = \frac{2^k \psi_k(v) - \psi_k(v - 1)}{2^k \psi_k(v) - \psi_k(v - 1)} = \frac{k}{r} \left| \mathcal{H}_v^{(k)} \right|^2$. Consequently

\[ \text{sgn} \delta_v = \text{sgn} (\varphi(v) - A) \]  

(3)

Note that $\varphi(v) \to 0$ as $v \to \infty$, and for all $v \geq r$, $\varphi(v + 1) < \varphi(v)$.

If for a given $0 \leq A \leq \frac{\left| \mathcal{H}_{v_0}^{(k)} \right|^2 - \left| \mathcal{H}_{v_0}^{(r)} \right|^2}{\left| \mathcal{H}_{v_0}^{(v)} \right|^2}$ the value $v_0$ is such that

$\varphi(v_0 + 1) \leq A \leq \varphi(v_0)$, then for $v \leq v_0$ we have, taking into account the equality (3) that $\delta_v \geq 0$. and consequently $\psi(A, r) \leq \psi(A, r + 1) \leq \ldots \leq \psi(A, v_0)$. In the case $v > v_0$ we have $\delta_v \leq 0$ and consequently $\psi(A, v_0) \geq \psi(A, v_0) \geq \ldots$. Therefore $\sup_{v \geq r} \psi(A, v) = \psi(A, v_0)$, if
Theorem 2

Let \( \varphi(v_0 + 1) \leq A \leq \varphi(v_0) \). Thus inequality (2) is proved.

It is clear that the inequality (2) is impossible if \( A > \frac{|T_v^{(k)}|^2 - |T_{v_0}^{(k)}|^2}{|T_v^{(r)}|^2} \),

which can be obtained for \( x(t) = T_{v_0}(t) \).

Given \( A \leq \frac{|T_v^{(k)}|^2 - |T_{v_0}^{(k)}|^2}{|T_v^{(r)}|^2} \), the constant \( \psi(A, v_0) \) is the best possible because the inequality becomes equality for \( x(t) = T_v(t) \).

Analogously to theorem 1 we can now prove for polynomials \( x \in \pi_n \) the following theorem

**Theorem 2** Let \( k, r, n \in N, k < r < n \). Then for any \( 0 \leq A \leq \frac{|T_v^{(k)}|^2}{|T_v^{(r)}|^2} \), and any \( x \in \pi_n \)

\[
\|x^{(k)}\|^2 \leq A \|x^{(r)}\|^2 + \sup_{r \leq v \leq n} \psi(A, v) \|x\|^2
\]

\[
= A \|x^{(r)}\|^2 + \frac{k}{r} \left( \left\|T_v^{(k)}\right\|^2 - \left\|T_v^{(r)}\right\|^2 \right) \|x\|^2
\]

(4)

if \( 0 \leq A \leq \frac{k}{r} \left( \left\|T_v^{(k)}\right\|^2 - \left\|T_v^{(r)}\right\|^2 \right) = \varphi(r) \). If \( \varphi(v_0 + 1) \leq A \leq \varphi(v_0) \), \( r \leq v_0 \leq n - 1 \), then

\[
\|x^{(k)}\|^2 \leq A \|x^{(r)}\|^2 + \psi(A, v_0) \|x\|^2.
\]

(5)

For a given \( 0 \leq A \leq \frac{|T_v^{(k)}|^2}{|T_v^{(r)}|^2} \) the constants at \( \|x\|^2 \) in (4) and (5) are the best possible.

**Proof** for \( x \in \pi_n \)

\[
\|x^{(k)}\|^2 = \sum_{k \leq v \leq n} c_v^2(x) \left\|T_v^{(k)}\right\|^2
\]

\[
\leq \frac{\left\|T_v^{(k)}\right\|^2}{\left\|T_v^{(r)}\right\|^2} \left\|x^{(r)}\right\|^2
\]
and consequently if \( A \geq \frac{\| \mathcal{P}_v^{(k)} \|_2^2 \| \mathcal{P}_{v-1}^{(k)} \|_2^2}{\| \mathcal{P}_v^{(r)} \|_2^2} \) then the inequality (1) holds with such \( A \) and \( B = 0 \). Therefore we will suppose that \( 0 \leq A \leq \frac{\| \mathcal{P}_v^{(k)} \|_2^2}{\| \mathcal{P}_v^{(r)} \|_2^2} \). For such an \( A \), we find as above

\[
\| x^{(k)} \|_2^2 \leq A \| x^{(r)} \|_2^2 + \sup_{v \geq n} \psi(A, v) \| x \|_2^2.
\]

Note that Bojanov and Varma’s result coincides with the inequality (4).

Denote by \( L_{r, 2, e^{-t^2}}^r(R, n) \) the set of functions \( x \in L_{r, 2, e^{-t^2}}^r(R) \) such that \( c_v(x) = 0 \) for \( v = k, \ldots, n - 1 \). For \( x \in L_{r, 2, e^{-t^2}}^r(R, n) \), we have

\[
\| x^{(k)} \|_2^2 = \sum_{v \geq n} c_v^2(x) \| \mathcal{P}_v^{(k)} \|_2^2 = \frac{\| \mathcal{P}_n^{(k)} \|_2^2 \| x^{(r)} \|_2^2}{\| \mathcal{P}_n^{(r)} \|_2^2} \| x^{(r)} \|_2^2.
\]

This inequality shows us that inequality (1) holds for any function \( x \in L_{r, 2, e^{-t^2}}^r(R, n) \) with \( B = 0 \) and \( A \geq \frac{\| \mathcal{P}_n^{(k)} \|_2^2}{\| \mathcal{P}_n^{(r)} \|_2^2} \).

Repeating the proof of Theorem 1 we obtain that for any \( k, r, n \in N, 0 < k < r < n \), any \( x \in L_{r, 2, e^{-t^2}}^r(R, n) \) and any \( 0 \leq A \leq \frac{\| \mathcal{P}_n^{(k)} \|_2^2}{\| \mathcal{P}_n^{(r)} \|_2^2} \)

\[
\| x^{(k)} \|_2^2 \leq A \| x^{(r)} \|_2^2 + \sup_{v \geq n} \psi(A, v) \| x \|_2^2.
\]

It’s easy to verify that

\[
\sup_{v \geq n} \psi(A, v) = \psi(A, v_0)
\]

if \( 0 \leq A \leq \varphi(n) \), and

\[
\sup_{v \geq n} \psi(A, v) = \psi(A, n)
\]

if \( \varphi(v_0 + 1) \leq A \leq \varphi(v_0), v_0 \geq n + 1 \).

Therefore we have proved the following
Theorem 3 Let $k, r, n \in N$, $0 < k < r < n$. Then for any $x \in L^r_{2, 2, e^{-t^2}}(R, n)$

$$\|x^{(k)}\|^2 \leq A\|x^{(r)}\|^2 + \psi(A, v_0) \|x\|^2$$

if $0 \leq A \leq \varphi(n)$, and

$$\|x^{(k)}\|^2 \leq A\|x^{(r)}\|^2 + \psi(A, n) \|x\|^2$$

if $\varphi(v_0 + 1) \leq A \leq \varphi(v_0)$, $v \geq n + 1$.

For a given $0 \leq A \leq \frac{R_n^{(k)}}{R_n^{(r)}}$ the constants at $\|x^{(r)}\|^2$ in these inequalities are the best possible.

References


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