Some Weak Separation Axioms in Bitopological Spaces

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Abstract

In this paper, we offer some new separation axioms called ultra-$R_{YS}$, ultra-$R_Y$, ultra-$R_D$ and ultra-$T_{YS}$. Moreover we study some of their basic properties.

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1 Introduction

Aull and Thron[2] introduced and studied extensively various separation axioms in between $T_0$ and $T_1$ in 1962. Shanin [15] introduced the notion of $R_0$ topological space in 1943. A topological space is $R_0$ if every open set contains
the closure of each of its singletons. Later, Dube [3] rediscovered it and studied some properties of this weak separation axiom. Among others, he showed that a topological space is $T_1$ if and only if it is $T_0$ and $R_0$. Several topologists (e.g. [2], [6], [7], [8], [9]) further investigated properties of $R_0$ topological spaces and many interesting results have been obtained. Also in 1974, Dube [3] introduced some more separation axioms $R_Y$, $R_{YS}$, $R_D$ and $R_{DD}$, which are weaker than $R_0$. The fact that if $X$ is not $R_0$, then there are some $x \in X$ such that $(1,2)\ker(\{x\}) \setminus (1,2)\ker(\{x\}) \neq \phi$ and also there are some $x \in X$ such that $(1,2)\ker(\{x\}) \setminus (1,2)\ker(\{x\}) \neq \phi$ suggests the Definition of $R_T$ spaces [13]. In this paper we generalize all the above axioms in bitopological spaces introduced in [10]. We define ultra-$R_T$, ultra-$R_D$, ultra-$R_{YS}$ and ultra-$T_{YS}$ and establish the relationship that ultra-$R_0$ $\Rightarrow$ ultra-$R_T$ $\Rightarrow$ ultra-$R_D$ $\Rightarrow$ ultra-$R_{YS}$ $\Rightarrow$ ultra-$T_{YS}$.

2 Preliminary Notes

Definition 2.1 A set $A$ of a topological space $(X,\tau)$ is called $\alpha$-open set [14] if $A \subset \text{int}(\text{cl}(\text{int}(A)))$ where $\text{int}(A)$ and $\text{cl}(A)$ denotes the interior and closure of $A$ with respect to $\tau$.

Definition 2.2 A topological space $X$ is said to be $R_0$ [3] if for each $x \in G$, $\text{cl}(\{x\}) \subset G$ where $G$ is an open set in $X$.

Definition 2.3 A topological space $X$ is said to be

(i) $R_{YS}$-space [16] if for $x, y \in X, \text{cl}(\{x\}) \neq \text{cl}(\{y\})$ implies $\text{cl}(\{x\}) \cap \text{cl}(\{y\}) = \phi$ or $\{x\}$ or $\{y\}$.

(ii) $R_D$-space [16] if for $x \in X, \text{cl}(\{x\}) \cap \text{ker}(\{x\}) = \{x\}$ implies that $d(\{x\}) = \text{cl}(\{x\}) \setminus \{x\}$ is closed, where $\text{ker}(\{x\}) = \cap\{G \in \tau \text{ and } x \in G\}$.

(iii) $R_T$-space [13] if for each $x \in X$, both $\text{ker}(\{x\}) \setminus \text{cl}(\{x\})$ and $\text{cl}(\{x\}) \setminus \text{ker}(\{x\})$ are degenerate. By a degenerate set we mean a null set or a singleton set.

In what follows, by a space $X$ we mean $(X,\tau_1,\tau_2)$, where $X$ is a nonempty set, $\tau_1$ and $\tau_2$ are topologies on $X$.

Definition 2.4 A subset $A$ of a space $X$ is called $(1,2)\alpha$-open [10] if $A \subset \tau_1\text{-int}(\tau_2\text{-cl}(\tau_1\text{-int}(A)))$, where a subset $A$ of $X$ is $\tau_1\tau_2$-open if $A \subset \tau_1 \cup \tau_2$ and it is $\tau_1\tau_2$-closed if its complement is $\tau_1\tau_2$-open, and the intersection of all the $\tau_1\tau_2$-closed sets containing $A$ is denoted by $\tau_1\tau_2\text{-cl}(A)$. The family of all $(1,2)\alpha$-open sets in $X$ is denoted by $(1,2)\alpha\mathcal{O}(X)$. The complement of a $(1,2)\alpha$-open set is called a $(1,2)\alpha$-closed set. The intersection of all $(1,2)\alpha$-closed (resp. $(1,2)\alpha$-open) sets containing $A$ is denoted by $(1,2)\alpha\text{cl}(A)$ (resp. $(1,2)\alpha\text{ker}(A)[8]$). The family of all $(1,2)\alpha$-open sets is denoted by $(1,2)\alpha\mathcal{O}(X)$.
Definition 2.5 A bitopological space $X$ is called ultra-$R_0$ (briefly $U_{R_0}$) [12] if for each $x \in G$, $(1,2)\alpha cl\{x\} \subseteq G$ where $G$ is a $(1,2)\alpha$-open set.

The following results were obtained in [12].

Theorem 2.6 Let $X$ be a space and $x,y \in X$. Then the following statements are satisfied

(i) If $x \in (1,2)\alpha cl\{y\}$, then $(1,2)\alpha cl\{x\} \subseteq (1,2)\alpha cl\{y\}$.

(ii) If $x \in (1,2)\alpha cl\{y\}$, then $y \in (1,2)\alpha ker\{x\}$.

(iii) If $X$ is $U_{R_0}$, then for each $x \in X$, $(1,2)\alpha cl\{x\} = (1,2)\alpha ker\{x\}$.

(iv) If $X$ is $U_{R_0}$, and $x \in (1,2)\alpha cl\{y\}$, then $y \in (1,2)\alpha cl\{x\}$ for any $x,y \in X$.

3 Some new separation axioms

Definition 3.1 A space $X$ is called

(i) ultra-$R_D$ (briefly $U_{R_D}$) if for each $x \in X$, $(1,2)\alpha cl\{x\} \cap (1,2)\alpha ker\{x\} = \{x\}$ implies that the $(1,2)\alpha$-derived set, $(1,2)\alpha d\{x\} = (1,2)\alpha cl\{x\} \setminus \{x\}$ is $(1,2)\alpha$-closed.

(ii) ultra-$R_T$ (briefly $U_{R_T}$) if for each $x \in X$, both $(1,2)\alpha ker\{x\} \setminus (1,2)\alpha cl\{x\}$ and $(1,2)\alpha cl\{x\} \setminus (1,2)\alpha ker\{x\}$ are degenerate.

(iii) ultra-$R_{YS}$ (briefly $U_{R_{YS}}$) if for $x,y \in X$, $(1,2)\alpha cl\{x\} \neq (1,2)\alpha cl\{y\}$ implies $(1,2)\alpha cl\{x\} \cap (1,2)\alpha cl\{y\} = \phi$ or $\{x\}$ or $\{y\}$.

(iv) ultra-$T_{YS}$ (briefly $U_{T_{YS}}$) if for $x \neq y$ implies $(1,2)\alpha cl\{x\} \cap (1,2)\alpha cl\{y\} = \phi$ or $\{x\}$ or $\{y\}$.

(v) ultra-$R_Y$ (briefly $U_{R_Y}$) if for all $x,y \in X$, $(1,2)\alpha cl\{x\} \cap (1,2)\alpha cl\{y\}$ is a degenerate set.

Remark 3.2 Obviously $U_{R_0}$ implies $U_{R_T}$. But the converse is not always true as it is shown by the following Example.

Example 3.3 Let $X = \{a,b,c,d\}$, $\tau_1 = \{\phi, X, \{a\}, \{a,b\}, \{c,d\}, \{a,c,d\}\}$, $\tau_2 = \{\phi, X, \{b\}\}$ and $(1,2)\alpha O(X) = \{\phi, X, \{a\}, \{a,b\}, \{c,d\}, \{a,c,d\}\}$. Here $X$ is $U_{R_T}$ but not $U_{R_0}$.

Theorem 3.4 Every space which is $U_{R_T}$, it is also $U_{R_D}$.
**Proof:** Let $X$ be ultra-$R_T$. Then both $(1,2)\alpha \ker \{ \{ x \} \} \setminus (1,2)\alpha \cl \{ \{ x \} \}$ and $(1,2)\alpha \cl \{ \{ x \} \} \setminus (1,2)\alpha \ker \{ \{ x \} \}$ are degenerate. Now let $< x > = (1,2)\alpha \cl \{ \{ x \} \} \cap (1,2)\alpha \ker \{ \{ x \} \}$. Then $(1,2)\alpha \ker \{ \{ x \} \} = < x > \cup D$ and $(1,2)\alpha \cl \{ \{ x \} \} = < x > \cup E$, where $D$ is not a subset of $(1,2)\alpha \cl \{ \{ x \} \}$ and $E$ is not a subset of $(1,2)\alpha \ker \{ \{ x \} \}$. Observe that $D$ and $E$ are degenerate sets. If $< x > = \{ x \}$, then $(1,2)\alpha \cl \{ \{ x \} \} = E \cup \{ x \}$ and $(1,2)\alpha \ker \{ \{ x \} \} = D \cup \{ x \}$. We need to prove that $(1,2)\alpha \cl \{ \{ x \} \} = (1,2)\alpha \cl \{ \{ x \} \} \setminus \{ x \}$ is $(1,2)\alpha$-closed. Let $U$ be a $(1,2)\alpha$-open set containing $(1,2)\alpha \ker \{ \{ x \} \}$. Then $(X \setminus U)$ is $(1,2)\alpha$-closed set. Hence $(X \setminus U) \cap (1,2)\alpha \cl \{ \{ x \} \} = E$ or $\phi$.

**Case (i)** If $(X \setminus U) \cap (1,2)\alpha \cl \{ \{ x \} \} = E$, then $E$ is the intersection two $(1,2)\alpha$-closed sets, hence $(1,2)\alpha$-closed.

**Case (ii)** $(X \setminus U) \cap (1,2)\alpha \cl \{ \{ x \} \} = \phi$, then $(1,2)\alpha \cl \{ \{ x \} \} \subset U$, $E \subset U$. Since $E$ is not a subset of $(1,2)\alpha \ker \{ \{ x \} \}$, there is a $(1,2)\alpha$-open set $V$ such that $x \in V$ and $E$ is not a subset of $V$. Then $(1,2)\alpha \cl \{ \{ x \} \} \cap (X \setminus V) = E$ is a $(1,2)\alpha$-closed set. Hence $X$ is $u R_D$.

**Remark 3.5** The converse of Theorem 3.4 is not always true as it is shown by the following Example.

**Example 3.6** $X = \{ a, b, c \}$, $\tau_1 = \{ \phi, X, \{ a \} \}$, $\tau_2 = \{ \phi, X, \{ a, c \} \}$ and $(1,2)\alpha O \{ \{ x \} \} = \{ \phi, X, \{ a \}, \{ a, b \}, \{ a, c \} \}$. Here $X$ is $u R_D$ but as $(1,2)\alpha \cl \{ \{ a \} \} \setminus (1,2)\alpha \ker \{ \{ a \} \} = \{ b, c \}$, $X$ is not $u R_T$.

**Theorem 3.7** Every $u R_T$ space is $u R_Y S$.

**Proof:** Let $X$ be $u R_T$ and $x, y \in X$. If $(1,2)\alpha \cl \{ \{ x \} \} \neq (1,2)\alpha \cl \{ \{ y \} \}$ and $(1,2)\alpha \cl \{ \{ x \} \} \cap (1,2)\alpha \cl \{ \{ y \} \} \neq \phi$. Hence assume that there exits an element $a \in X$ such that $a \neq x, a \neq y$ and $a \in (1,2)\alpha \cl \{ \{ x \} \} \cap (1,2)\alpha \cl \{ \{ y \} \}$, then $a \in (1,2)\alpha \cl \{ \{ x \} \}$ and $a \in (1,2)\alpha \cl \{ \{ y \} \}$. So $x, y \in (1,2)\alpha \ker \{ \{ a \} \}[12]$. As $X$ is $u R_T$ $(1,2)\alpha \ker \{ \{ a \} \} = < a > \cup E$ where $E$ is a degenerate set and $E$ is not a subset of $(1,2)\alpha \cl \{ \{ a \} \}$. There exits four possible cases if $x \in (1,2)\alpha \ker \{ \{ a \} \}$ and $y \in (1,2)\alpha \ker \{ \{ a \} \}$.

**Case (i)** Assume $x \in < a >$ and $y \in < a >$. Then $x \in (1,2)\alpha \cl \{ \{ a \} \}$ and $y \in (1,2)\alpha \cl \{ \{ a \} \}$. Hence $(1,2)\alpha \cl \{ \{ x \} \} = (1,2)\alpha \cl \{ \{ a \} \} = (1,2)\alpha \cl \{ \{ y \} \}$, a contradiction.

**Case (ii)** Assume $\{ x \} = E$ and $y \in < a >$. So $\{ x \} \notin (1,2)\alpha \cl \{ \{ a \} \}$ and $y \in (1,2)\alpha \cl \{ \{ a \} \}$. Then $(1,2)\alpha \cl \{ \{ y \} \} = (1,2)\alpha \cl \{ \{ a \} \}}$. Here also we have two cases to discuss.

**Case (a)** Let $y \in (1,2)\alpha \cl \{ \{ x \} \}$ and by assumption $x \notin (1,2)\alpha \cl \{ \{ a \} \}$. So $x \in X \setminus (1,2)\alpha \cl \{ \{ a \} \}$, where $X \setminus (1,2)\alpha \cl \{ \{ a \} \}$ is a $(1,2)\alpha$-open set containing the point $x$. Hence $(1,2)\alpha \ker \{ \{ x \} \} \subset X \setminus (1,2)\alpha \cl \{ \{ a \} \}$. Then $(1,2)\alpha \cl \{ \{ x \} \} \setminus (1,2)\alpha \ker \{ \{ x \} \} \supset (1,2)\alpha \cl \{ \{ a \} \} \supset \{ y, a \}$, implies that $(1,2)\alpha \cl \{ \{ x \} \} \setminus (1,2)\alpha \ker \{ \{ x \} \}$ is not a degenerate set. A contradiction.
Case (b) Let \( y \notin (1,2)\alpha cl\{x\} \). Since \( y \in (1,2)\alpha cl\{x\} \) and \( a \in (1,2)\alpha cl\{x\} \), we have \( y \in (1,2)\alpha cl\{x\} \), again a contradiction.

Case(iii) Let \( x \in < a > \) and \( \{y\} = E \). Its proof is similar to Case (ii).

Case (iv) Let \( \{x\} = \{y\} = E \). Then \( (1,2)\alpha cl\{x\} = (1,2)\alpha cl\{y\} \), a contradiction. Hence if \( (1,2)\alpha cl\{x\} \neq (1,2)\alpha cl\{y\} \), then \( (1,2)\alpha cl\{x\} \cap (1,2)\alpha cl\{y\} \) = \( \phi \) or \( \{x\} \) or \( \{y\} \). So \( X \) is \( U_{Y_{YS}} \).

The converse of the above Theorem is not always true. This can be shown by the Example 3.13.

**Theorem 3.8** Every \( U_{Y_{YS}} \) space is \( U_{Y_{D}} \), but not the converse.

**Proof:** Let \( X \) be a \( U_{Y_{YS}} \) space. There are three cases to discuss.

Case (i) Let \( (1,2)\alpha cl\{x\} \cap (1,2)\alpha cl\{y\} \) = \( \{x\} \). Then \( (1,2)\alpha cl\{x\} = \phi \) and hence \( (1,2)\alpha \)-closed set.

Case (ii) Let \( (1,2)\alpha cl\{x\} \cap (1,2)\alpha cl\{y\} \) = \( \{y\} \). Then \( (1,2)\alpha cl\{y\} = \phi \).

Case(iii) Let \( (1,2)\alpha cl\{x\} \cap (1,2)\alpha cl\{y\} \) = \( \phi \) and by assumption, \( (1,2)\alpha cl\{x\} \cap (1,2)\alpha ker\{x\} \) = \( \{x\} \). Hence we get a contradiction. Therefore \( X \) is \( U_{Y_{D}} \).

**Example 3.9** The converse of Theorem 3.8 is not always true as it is shown in this Example. Let \( X = \{a,b,c,d\} \), \( \tau_1 = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\} \), \( \tau_2 = \{\phi, X, \{a\}, \{d\}\} \) and \( (1,2)\alpha O\{X\} = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}\} \). This space \( X \) is \( U_{Y_{D}} \) but not \( U_{Y_{YS}} \).

**Remark 3.10** Obviously every \( U_{Y_{YS}} \) space is \( U_{Y_{Y}} \), but the converse is not true. Since the space in Example 3.9 is a \( U_{Y_{Y}} \) space but not a \( U_{Y_{YS}} \) space.

**Definition 3.11** A space \( X \) is said to be ultra-\( Y_{YS} \) (briefly \( U_{Y_{YS}} \)) space if for all \( x, y \in X \), \( x \neq y, (1,2)\alpha cl\{x\} \cap (1,2)\alpha cl\{y\} = \phi \) or \( \{x\} \) or \( \{y\} \).

**Remark 3.12** Every \( U_{Y_{YS}} \) space is \( U_{Y_{YS}} \), but not the conversely.

**Example 3.13** This Example gives a space which is \( U_{Y_{YS}} \) but not \( U_{Y_{YS}} \). Let \( X = \{a,b,c,d\} \), \( \tau_1 = \{\phi, X, \{a\}, \{b\}\} \), \( \tau_2 = \{\phi, X, \{b,c\}\} \) and \( (1,2)\alpha O\{X\} = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\} \).

**Definition 3.14** Let \( X \) be a bitopological space and two points \( x, y \in X \) are said to be \( (1,2)\alpha \)-separated iff \( \{x\} \cap (1,2)\alpha cl\{y\} = \phi \).

**Theorem 3.15** For a space \( X \), the following are equivalent.

(i) \( X \) is \( U_{Y_{YS}} \).

(ii) For any two distinct points \( x, y \in X \) either of the points has an empty \( (1,2)\alpha \)-derived set ((1,2)\alpha \-derived-set) or \( (1,2)\alpha cl\{x\} \cap (1,2)\alpha cl\{y\} = \phi \).
(iii). The (1,2)$\alpha$-closure of (1,2)$\alpha$-derived sets any two distinct points are dis-
joint.

(iv). The (1,2)$\alpha$-derived sets of any two distinct points are (1,2)$\alpha$-separated

**Proof:** (i) ⇒ (ii)

Let $X$ is $\mathcal{U}T_{YS}$ and if for any $x,y \in X,(1,2)\text{acl}(\{x\}) \cap (1,2)\text{acl}(\{y\}) = \phi$ nothing
to prove. If not let (1,2)$\text{acl}(\{x\}) \cap (1,2)\text{acl}(\{y\}) = \{x\}$, then (1,2)$\text{ad}(\{x\}) = \phi$. 

(ii) ⇒ (iii)

Let $(1,2)\text{acl}((1,2)\text{ad}(\{x\})) \cap (1,2)\text{acl}((1,2)\text{ad}(\{y\})) = \phi$. If any one of the
derived set is $\phi$. Then nothing to prove. Now $(1,2)\text{acl}((1,2)\text{ad}(\{x\})) \subset 
(1,2)\text{acl}(\text{ad}(\{x\}))$, and hence $(1,2)\text{acl}((1,2)\text{ad}(\{x\}) \cap (1,2)\text{acl}((1,2)\text{ad}(\{y\})) \subset 
(1,2)\text{acl}(\{x\}) \cap (1,2)\text{acl}(\{y\}) = \phi$. Hence the result.

(iii)⇒(iv) Obvious.

(iv)⇒(i) Let $x,y \in X$ and $x \neq y$ such that $(1,2)\text{acl}((1,2)\text{ad}(\{x\})) \cap (1,2)\text{ad}(\{y\})
= \phi$. There are two possibilities, $(1,2)\text{acl}((1,2)\text{ad}(\{x\})) = (1,2)\text{ad}(\{x\})$ or 
$(1,2)\text{acl}((1,2)\text{ad}(\{x\})) = (1,2)\text{acl}(\{x\})$.

**Case (i)** Let $(1,2)\text{ad}(\{x\})$ is (1,2)$\alpha$-closed for each $x \in X$.

Then $(1,2)\text{ad}(\{x\}) \cap (1,2)\text{ad}(\{y\}) = \phi$ or $(1,2)\text{ad}(\{x\}) \cap (1,2)\text{ad}(\{y\}) = \phi$ or $y$.

If it is $y$ then $y$ is (1,2)$\alpha$-closed and so $(1,2)\text{acl}(\{x\}) \cap (1,2)\text{acl}(\{y\}) = \{y\}$. If it
is $\phi$, then $(1,2)\text{acl}(\{x\}) \cap (1,2)\text{acl}(\{y\}) = \phi$ or $\{x\}$.

**Case (ii)** Let $(1,2)\text{acl}((1,2)\text{ad}(\{x\})) = (1,2)\text{acl}(\{x\})$, then we have
$(1,2)\text{acl}(\{x\}) \cap (1,2)\text{ad}(\{y\}) = \phi$ and hence $(1,2)\text{acl}(\{x\}) \cap (1,2)\text{ad}(\{y\}) = \phi$ or 
$\{y\}$.

**Remark 3.16** The spaces $\mathcal{U}R_T$ and $\mathcal{U}T_{YS}$ are independent of each other
can be seen from Example 3.6 in which $X$ is $X$ is $\mathcal{U}T_{YS}$ but not $\mathcal{U}R_T$ and also
from the following Example.

**Example 3.17** Let $X = \{a,b,c,d\}, \tau_1 = \{\phi,X,\{a\},\{a,b\},\{c,d\},\{a,c,d\}\}$, $\tau_2 = 
\{\phi, X\}, (1,2)\alpha \mathcal{O}(X) = \tau_1$. This space $X$ is not $\mathcal{U}T_{YS}$ but it is $\mathcal{U}R_T$.

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Some weak separation axioms


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