Special Finsler Spaces Admitting
Metric Like Tensor Field

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Abstract

In this work we modify the special Finsler spaces like C-reducible, semi-C-reducible, quasi-C-reducible are admitting the tensor field \( X_{h k} = h_{h k} + X_{00} l_{h} l_{k} \), which satisfies the condition \( C_{i j} X_{h k} = C_{i j k} \). Similarly, we have also worked out for S3-like, \( C^{h} \)-recurrent, P-reducible and T-conditions of Finsler spaces.

Mathematics Subject Classification: 53C60;

Keywords: Finsler Space, C-reducible, Semi-C-reducible, C2-like, Quasi-C-Reducible, S3-Like, P-reducible, T-Condition

1 Introduction

The terminology and notations are referred to [1], [4] and [6]. Let \( F^{n} = (M^{n}, L) \) be a Finsler space on a differentiable manifold \( M \) endowed with a fundamental function \( L(x, y) \). We use the following notations: [4][6]

\[ a) \quad g_{i j} = \frac{1}{2} \dot{\partial}_{i} \dot{\partial}_{j} L^{2}, \quad \dot{\partial}_{i} = \frac{\partial}{\partial y^{i}} \]

\[ b) \quad C_{i j k} = \frac{1}{2} \dot{\partial}_{k} g_{i j} \]
c) \( h_{ij} = g_{ij} - l_il_j, \)

d) \( C^h_{hk} = C_k, C^h_l = 0, \)

e) \( h^m_k h_{mj} = h_{jk}, h^m_j l_m = 0, \)

\[ (1) \]

f) \( C^m_{hr} g_{mj} = C_{hjr}, \)

g) \( l^m_m = 0, p^m_i l_m = 0, \)

h) \( h^m_j X_{mk} = X_{jk} - X_{kl} l_j, \)

i) \( X_{lo} l_j = X_{jo} l_i, X_0 = X_{00} l_i. \)

There are three kinds of torsion tensors in carton’s theory of Finsler spaces. Two of them are \( h(h\nu)\)-torsion tensor \( C_{ijk} \) and \( (\nu)h\nu\)-torsion tensor \( P_{ijk} \), which are symmetric in all their indices. The contravariant components of \( (\nu)h\nu\)-torsion tensor is given by

\[ C^h_{ij} = g^h_{hk} C_{ijk}, \]

which may be treated as Christoffel symbols of second kind of each tangent Riemannian space of Finsler space \( F^n \). Here, \( g^h_{hk} \) is the inverse of metric tensor \( g_{hk} \) of \( F^n \). If \( l_i \) is the normalized element of support \( h_{ij} \) is the angular metric tensor given by

\[ h_{ij} = g_{ij} - l_il_j, \]

Then

\[ C^h_{ij} h_{hk} = C_{ijk}. \]

(2)

If \( b_i \) are components of a concurrent vector field, then \( b_i/j = -g_{ij} \) and \( b_{ij} = 0 \), where \( /j \) and \( |j \) denote the \( h \) and \( \nu \)-covariant derivatives with respect to cartons connection \( CT \). From this it follows that \( b_i \) are functions of position only, and \( C^h_{ij} b_h = 0 \). Thus if we consider a tensor field is given by \( B_{ij} = g_{ij} + \alpha l_il_j + \beta b_i b_j \), where \( \alpha \) and \( \beta \) are scalar functions, then

\[ C^h_{ij} B_{hk} = C_{ijk}. \]

(3)

The purpose of the present paper is to study the existence of any symmetric covariant tensor \( X_{hk} \) which satisfies

\[ X_{hk} = h_{hk} + X_{00} l_h l_k. \]

(4)

Throughout the paper we are concerned with non-Riemannian Finsler space having positive definite metric tensor \( g_{ij} \). From (4) we have,

\[ C^h X_{hk} = C_k \quad \text{and} \quad C^h_{ij} X_{h0} = 0, \]

(5)

where \( C^h = C^h_{ij} g^{ij} \) and \( 0 \) denotes the contraction with \( l^i \).

### 2 The Existence Of Covariant Tensor \( X_{hk} \) In C-Reducible Finsler Space:

In a C-reducible Finsler space the \( (h)h\nu\)-torsion tensor \( C^h_{ij} \) is given by,[2][5]

\[ C^h_{ij} = (C^h_{ij} + C^h_{ji} + C^h_{ij})/(n + 1). \]

(6)
Now contracting above equation by $X_{hk}$, then from equations (4) and (1)(d), we have

\[
\begin{align*}
C^h_{ij}X_{hk} &= X_{hk}(C^h_{hi} + C_i h^j_j + C_j h^h_h)/(n + 1), \\
C^h_{ij}X_{hk} &= 1/(n + 1)(C^h_{hi}(h_{hk} + X_{00l}l_k) + C_i h^j_j(h_{hk} + X_{00l}l_k) + C_j h^h_h(h_{hk} + X_{00l}l_k)) \\
C^h_{ij}X_{hk} &= 1/((n + 1))C_k h_{ij} + C_i h_{jk} + C_j h^h_h X_{00l}l_k + C_j h_{ik} + C_j h^h_h X_{00l}l_k \\
C^h_{ij}X_{hk} &= 1/(n + 1)(C_k h_{ij} + C_i h_{jk} + C_i h_{00l}l_k(l_j - l_j) + C_j h_{ik} + C_j X_{00l}l_k(l_i - l_i)) \\
C^h_{ij}X_{hk} &= 1/(n + 1)(C_k h_{ij} + C_i h_{jk} + C_j h_{ik}) \\
C^h_{ij}X_{hk} &= C_{ijh}. \tag{7}
\end{align*}
\]

**Theorem 2.1** In a C-reducible Finsler space the covariant tensor field $X_{hk}$ satisfies (4) is of the form (7).

Consider a Semi-C-reducible Finsler space $C^h_{ij}$ is given by [3],

\[
C^h_{ij} = (C^h_{hi} h_{ij} + C_i h^h + C_j h^h)/(n + 1) + (C^h_{i} C_{j} )q/C^2. \tag{8}
\]

Now contracting above equation by $X_{hk}$ and using equations (4) and (1)(d), we have

\[
\begin{align*}
C^h_{ij}X_{hk} &= X_{hk}(C^h_{hi} h_{ij} + C_i h^h + C_j h^h)/(n + 1) + X_{hk}(C^h_{i} C_{j})q/C^2, \\
C^h_{ij}X_{hk} &= (C^h_{hi} h_{hk} + X_{00l}l_k) + C_i h^h h_{hk} + X_{00l}l_k) + C_j h^h h_{hk} + X_{00l}l_k)) \\
&\quad p/(n + 1) + (C^h_{i} C_{j})h_{hk} + X_{00l}l_k)q/C^2, \\
C^h_{ij}X_{hk} &= (C_k h_{ij} + C_i h_{jk} + C_j h_{ik})p/(n + 1) + (C_k C_{i} C_{j})q/C^2, \\
C^h_{ij}X_{hk} &= C_{ijh}. \tag{9}
\end{align*}
\]

**Theorem 2.2** In a Semi-C-reducible Finsler space the tensor field $X_{hk}$ satisfies (4) is of the form (9).

Consider a Quasi-C-reducible Finsler space $C^h_{ij}$ is given by [3],

\[
C^h_{ij} = (C^h_{i} A_{j} + C_i A^h + C_j A^h). \tag{10}
\]

Now contracting above equation by $X_{hk}$ and using equations (4) and (1)(d), we have

\[
\begin{align*}
C^h_{ij}X_{hk} &= (C^h_{i} A_{j} + C_i A^h + C_j A^h)X_{hk}, \\
C^h_{ij}X_{hk} &= (C^h_{i} A_{j} h_{hk} + X_{00l}l_k) + C_i A^h h_{hk} + X_{00l}l_k(k) \\
&\quad + C_j A^h h_{hk} + X_{00l}l_k(k), \\
C^h_{ij}X_{hk} &= (C_k A_{ij} + C_i A_{jk} + C_j A_{ik}), \\
C^h_{ij}X_{hk} &= C_{ijh}. \tag{11}
\end{align*}
\]

**Theorem 2.3** In a Quasi-C-reducible Finsler space the tensor field $X_{hk}$ satisfies (4) is of the form (11) if $A^h h_{hk} = 0$. 

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**Special Finsler spaces**
3 The Existence Of Covariant Tensor $X_{hk}$ In S3-Like Finsler Space:

In a S3-like Finsler space, whose $\nu-$ curvature tensor of cartons connection $C$ is given by [5],

$$L^2S^m_{hk} = S(h_{ih}h^m_k - h_{ik}h^m_h).$$

(12)

Contacting above equation by $X_{mj}$ and using equations (4) and (1)(d), we have

\[
\begin{align*}
L^2S^m_{ihk}X_{mj} &= S(h_{ih}h^m_k X_{mj} - h_{ik}h^m_h X_{mj}), \\
L^2S^m_{ihk}X_{mj} &= S[h_{ih}h^m_k (h_{mj} - X_{00}l_m l_j) - h_{ik}h^m_h (h_{mj} - X_{00}l_m l_j)], \\
L^2S^m_{ihk}X_{mj} &= S[h_{ih}h_{jk} - X_{00}h_{ik}h^m_h l_m l_j - h_{ik}h_{hj} + X_{00}h_{ik}h^m_h l_m l_j], \\
L^2S^m_{ihk}X_{mj} &= S[h_{ih}h_{jk} - h_{ik}h_{hj}], \\
L^2S^m_{ihk}X_{mj} &= S_{hijk}. 
\end{align*}
\]

(13)

**Theorem 3.1** In a S3-like Finsler space, the covariant tensor field $X_{mj}$ satisfies (4) is of the form (13).

Next we consider S4-like Finsler space, whose $\nu$-curvature tensor of cartons connection $C$ is given by [7],

$$L^2S^m_{ihk} = h^m_i M_{ik} + h_{ik}M^m_h - h_{hk}M^m_i - h^m_i M_{hk}.$$

(14)

Contacting above equation by $X_{mj}$ and using equations (4) and (1)(d), we have

\[
\begin{align*}
L^2S^m_{ihk}X_{mj} &= h^m_i M_{ik}X_{mj} + h_{ik}M^m_h X_{mj} - h_{hk}M^m_i X_{mj} - h^m_i M_{hk}X_{mj}, \\
L^2S^m_{ihk}X_{mj} &= h_{hj}M_{ik} + h_{ik}M_{hj} - h_{hk}M_{ij} - h_{ij}M_{hk}, \\
L^2S^m_{ihk}X_{mj} &= L^2S_{hijk}. 
\end{align*}
\]

(15)

**Theorem 3.2** In a S4-like Finsler space, the covariant tensor field $X_{ij}$ satisfies (4) is of the form (15).

Next we consider a $\nu$-curvature tensor of Cartons connection $C$ is given by [2],

$$S^m_{ihk} = C^m_{hr}C^r_{ik} - C^m_{kr}C^r_{ih}.$$ 

(16)

Contacting above equation by $X_{mj}$ and using equations (4) and (1)(d), we have

\[
\begin{align*}
S^m_{ihk}X_{mj} &= C^m_{hr}C^r_{ik}X_{mj} - C^m_{kr}C^r_{ih}X_{mj}, \\
S^m_{ihk}X_{mj} &= C^m_{hr}C^r_{ik}(h_{mj} + X_{00}l_m l_j) - C^m_{kr}C^r_{ih}(h_{mj} + X_{00}l_m l_j), \\
S^m_{ihk}X_{mj} &= C_{hjr}C^r_{ik} - C_{krj}C^r_{ih}, \\
S^m_{ihk}X_{mj} &= S_{hijk}. 
\end{align*}
\]

(17)
Theorem 3.3 In a $\nu$-curvature tensor, the covariant tensor field $X_{mj}$ satisfies (4) is of the form (17).

Now we concerned with a space of scalar curvature in Berwald’s sense. It is characterized by the equation is [2]

$$R_{ij} = h^i_k k_j - h^i_j k_k.$$  

(18)

where $h_{ik}$ is the angular metric tensor and the scalar curvature $K$ is a function scalar field.

Contacting above equation by $X_{il}$ and using equations (4) and (1)(d), we get

$$R_{ij} X_{il} = h^i_k K_j X_{il} - h^i_j K_k X_{il},$$

$$R_{ij} X_{il} = h^i_k K_j h_{il} - h^i_j K_k X_{il},$$

$$R_{ij} X_{il} = K_j h_{ki} - K_k h_{jl},$$

$$R_{ij} X_{il} = R_{ijk}.$$  

(19)

Theorem 3.4 In a space of scalar curvature tensor, the covariant tensor field $X_{il}$ satisfies (4) is of the form (19).

4 The Existence Of Covariant Tensor $X_{hk}$ In P-Reducible Finsler Space:

The P-reducible Finsler space is given as [5],

$$P^m_{jk} = (h^m_{jk} P_k + h^m_{jk} P_m + h^m_{jk} P_j)/(n + 1),$$  \hspace{1cm} (20)

Contacting above equation by $X_{mi}$ and using equations (4) and (1)(d), we have

$$P^m_{jk} X_{mi} = (h^m_{jk} P_k X_{mi} + h^m_{jk} P^m_{jk} X_{mi} + h^m_{jk} P_j X_{mi})/(n + 1),$$

$$P^m_{jk} X_{mi} = (h^m_{jk} P_k (h_{mi} + X_{00} l_ml_l) + h^m_{jk} P^m_{jk} (h_{mi} + X_{00} l_ml_l) +$$

$$h^m_{jk} P^m_{jk} (h_{mi} + X_{00} l_ml_l))/(n + 1),$$

$$P^m_{jk} X_{mi} = (h_{ij} P_k + h_{jk} P_l + h_{ki} P_j)/(n + 1),$$

$$P^m_{jk} X_{mi} = P_{ijk}.$$  \hspace{1cm} (21)

Theorem 4.1 In a P-reducible Finsler space, the covariant tensor field $X_{mi}$ satisfies (4) is of the form (21).

5 The Existence Of Covariant Tensor $X_{hk}$ In $C^h$-Recurrent Finsler Space:

Now we consider a $C^h$-recurrent Finsler space is given as [2],

$$C^m_{jk/h} = \alpha C^m_{jk}.$$  \hspace{1cm} (22)
Contacting above equation by $X_{mi}$ and using equations (4) and (1)(d), we can written as
\[
\begin{align*}
C_{jk/h}^m X_{mi} &= \alpha_h C_{jk}^m X_{mi}, \\
C_{jk/h}^m X_{mi} &= \alpha_h C_{jk}^m (h_{mi} + X_{00l}l_i), \\
C_{jk/h}^m X_{mi} &= \alpha_h C_{ijk}, \\
C_{jk/h}^m X_{mi} &= C_{ijk/h}.
\end{align*}
\tag{23}
\]

**Theorem 5.1** In a $C^h$-recurrent Finsler space, the covariant tensor field $X_{mi}$ satisfies (4) is of the form (23).

### 6 The Existence Of Covariant Tensor $X_{hk}$ In T-Condition :

Finsler space satisfying T-condition can be defined as,[6]
\[
T^m_{ij} = L c^m_{ij} + l^m c_{ijk} + l_i c^m_{jk} + l_j c^m_{ik} + l_k c^m_{ij} = 0,
\tag{24}
\]
Contacting above equation by $X_{hm}$ and using equations (4) and (1)(d), we have

\[
\begin{align*}
T^m_{ijk} X_{hm} &= L C^m_{ij/k} X_{hm} + l^m C_{ijk} X_{hm} + l_i C^m_{jk} X_{hm} + l_j C^m_{ik} X_{hm} + l_k C^m_{ij} X_{hm} = 0, \\
T^m_{ijk} X_{hm} &= L C^m_{ij/k} X_{hm} + l^m C_{ijk} (h_{hm} + X_{00l}l_m) + l_i C^m_{jk} (h_{hm} + X_{00l}l_m) + \\
&\hspace{1cm} l_j C^m_{ik} (h_{hm} + X_{00l}l_m) + l_k C^m_{ij} (h_{hm} + X_{00l}l_m) = 0, \\
T^m_{ijk} X_{hm} &= L C_{ij/k} + l^m C_{ijk} + l_i C_{ijk} + l_j C_{ijk} + l_k C_{ijk} = 0, \\
T^m_{ijk} X_{hm} &= T_{ijk} + X_{00l}C_{ijk} = 0, \\
T^m_{ijk} X_{hm} &= T_{ijk} = 0.
\end{align*}
\tag{25}
\]

**Theorem 6.1** If the Finsler space satisfying T-condition, then the covariant tensor field $X_{hm}$ satisfies (4) is of the form (25) provided $X_{00l}C_{ijk} = 0$.

Finsler space satisfying generalized T-condition can be defined as [5],
\[
T^h_j = L C^h_{ij} + l^h C_j + l_j C^h = 0.
\tag{26}
\]
Contacting above equation by $X_{ih}$ and using equations (4) and (1)(d), we obtain

\[
\begin{align*}
T^h_j X_{ih} &= L C^h_{ij} X_{ih} + l^h C_j X_{ih} + l_j C^h X_{ih} = 0, \\
T^h_j X_{ih} &= L C^h_{ij} (h_{ih} + X_{00l}l_h) + l^h C_j (h_{ih} + X_{00l}l_h) + l_j C^h (h_{ih} + X_{00l}l_h) = 0, \\
T^h_j X_{ih} &= L C_{ij} + l_i C_j + l_j C_i = 0, \\
T^h_j X_{ih} &= T_{ij}.
\end{align*}
\tag{27}
\]
Theorem 6.2 If the Finsler space satisfying generalized T-Condition, then the covariant tensor field $X_{ih}$ satisfies (4) is of the form (27).

References


Received: September 24, 2008