Characterization of the Curves $Q(p, k, r, n)$ as Analogous of the Trinomial Arcs $G(p, k, r, n)$

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Abstract

The main purpose of the present note is to study the family of trinomial curves $Q(p, k, r, n)$ which is one of the solutions of the trinomial equation $z^n = \alpha z^k + (1 - \alpha)$ with $-\infty < \alpha < 0$. We prove that these curves are outside the unit disk. Moreover, using the descriptive claim 1 of Fell [3], we succeed in sketching this family of curves. Afterwards, we realize that an analogy exists between these curves $Q(p, k, r, n)$ and the trinomial arcs $G(p, k, r, n)$ defined in [4] as one of the solutions of the trinomial equation in the case $0 < \alpha < 1$. This analogy is proved in this paper.

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1 Preliminaries

Since the beginning of the previous century, the study of the trinomial equations was one of the most important subjects of investigations. This theme was discussed in several manuscripts and by many authors, but, the first description of the form and the location of the trajectories of zeros of these equations was fulfilled by Fell [3]. There are many types of trinomial arcs, i.e. these trajectories of roots of the trinomial equation $z^n = \alpha z^k + (1 - \alpha)$. In [4], one of these categories corresponding to the case $0 < \alpha < 1$ denoted by $G(p, k, r, n)$ was studied. At first, the aim of our work was to study one family of trinomial curves which we denote by $Q(p, k, r, n)$ and which corresponds to the case $-\infty < \alpha < 0$. After that, we realized that these curves $Q(p, k, r, n)$ are analogous to the trinomial arcs $G(p, k, r, n)$ which we studied in [4]. We
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succeeded as it happened in proving that an analogy exists between these two families of arcs.

When $\alpha = 0$, the trinomial equation possess $n$ roots, the $n^{th}$ roots of unity. When $\alpha$ varies from 0 to $-\infty$, the trajectories of the $n$ roots are continuous curves. To describe these trajectories, Fell [3] used the following terminology:

\[ A = \{ k^{th} \text{ roots of unity} \}, \]
\[ B = \{(n - k)^{th} \text{ roots of } -1 \} \]
\[ C = \{ n^{th} \text{ roots of unity} \}. \]

Assume that $\gamma \in C$ and that $\beta$ is the unique nearest neighbor of $\gamma$ in $A \cup B$. Fell [3] considered three cases, but in this work we are interested in the first case, i.e. $\beta \in B$ and $\beta \notin A$. When $\alpha$ moves from 0 to $-\infty$, the trajectories starting at $\gamma$ is a continuous curve such that $\rho$ varies from 1 to $+\infty$. This trajectory is tangent to the line segment $\theta = \arg(\beta)$.

Let us recall that an angle $\theta$ is said feasible for the trinomial equation with $-\infty < \alpha < 0$ if $\text{sign}(\sin(n \theta)) = -\text{sign}(\sin k \theta) = \text{sign}(\sin(n - k) \theta)$. Fell [3] proved that in the case $\beta \in B$, $\beta \notin A$ and $-\infty < \alpha < 0$, the feasible angles $\theta$ belong to intervals of length less than or equal to $\pi/n$ and bounded by $\arg(\beta)$ and $\arg(\gamma)$, such that $\beta$ is an $(n - k)^{th}$ root of $-1$ and $\gamma$ is an $n^{th}$ root of unity.

In the present note, we deal with these trinomial curves for which the feasible angles $\theta$ belong to an interval of the form $[(2p + 1)\pi/(n - k), 2\pi r/n]$, where $p$ and $r$ are two integers. One can easily see that $\beta$ such that $\arg(\beta) = (2p + 1)\pi/(n - k)$ is an $(n - k)^{th}$ root of $-1$ and $\gamma$ such that $\arg(\gamma) = 2\pi r/n$ is an $n^{th}$ root of unity. We denote these trinomial curves by $Q(p, k, r, n)$.

Let us note that under the symmetry map $z \mapsto \overline{z}$, the upper and lower half-planes are symmetrical. Thus, it’s sufficient to restrict the study of the curves $Q(p, k, r, n)$ to the upper half-plane.

2 Trinomial curves $Q(p, k, r, n)$

The trajectories of roots of the trinomial equation in the case $-\infty < \alpha < 0$ when $n = 2$ are linear. Hence, we define the family of trinomial curves $Q(p, k, r, n)$ in the following manner.

Suppose that $n$ is an integer larger than or equal to 3. The trinomial curve $Q(p, k, r, n)$ is the set of roots of the trinomial equation $z^n = \alpha z^k + (1 - \alpha)$ with $z$ is a complex variable, $\alpha$ is a real number such that $-\infty < \alpha < 0$ and the feasible angles belong to the interval $[(2p + 1)\pi/(n - k), 2\pi r/n]$, where $p$ and $r$ are two integers such that $r \geq p + 1$ and $k$ is an integer satisfying $2(r - p - 1)n/(2r - 1) < k < [2(r - p) - 1]n/2r$. 
Characterization of the curves $Q(p,k,r,n)$

Now, we state the following result proving the existence of the trinomial curves $Q(p,k,r,n)$.

**Proposition 2.1** Assume that $n$ is an integer larger than or equal to 3, $k$ is an integer such that $0 < k < n$ and $-\infty < \alpha < 0$. Considering the trinomial equation $z^n = \alpha z^k + (1 - \alpha)$ with $2(r-p-1)n/(2r-1) < k < [2(r-p-1)n/2r$, where $p$ and $r$ are two integers satisfying $r \geq p+1$, any angle of the interval $[(2p+1)\pi/(n-k), 2\pi r/n]$ is feasible.

**Proof.** Let us consider an integer $k$ such that $2(r-p-1)n/(2r-1) < k < [2(r-p-1)n/2r$. As $0 < k < n$, the integers $p$ and $r$ verify $2(r-p-1)/(2r-1) \geq 0$, i.e. $r \geq p+1$. Now, let $\theta$ be an angle such that $(2p+1)\pi/(n-k) < \theta < 2\pi r/n$. One can have $(2p+1)\pi n/(n-k) < n\theta < 2\pi r$. Since $2(r-p-1)n/(2r-1) < k$, so $(2r-1)\pi < (2p+1)\pi n/(n-k)$. This means that $\sin n\theta < 0$. Moreover, we can find that $(2p+1)\pi < (n-k)\theta < 2\pi r(1-k/n)$. Because $r \geq p+1$, then $(r-p-1)n/r \leq 2(r-p-1)n/(2r-1) < k$. It yields that $2\pi r(1-k/n) < 2(p+1)\pi$ and that $\sin n\theta < 0$. In addition, one can obtain that $(2p+1)\pi k/(n-k) < k\theta < 2\pi rk/n$. Remarking that $2(r-p-1)\pi < (2p+1)\pi k/(n-k)$ and that $2\pi rk/n < [2(r-p-1)\pi$, we deduce that $\sin k\theta > 0$. Thus, the angle $\theta$ is feasible as we proved that $\sin n\theta = -\sin k\theta = \sin (n-k)\theta$.

**Corollary 2.2** Suppose that $Q(p,k,r,n)$ is a trinomial curve. For any feasible angle $\theta$, we have $\sin n\theta < 0$, $\sin k\theta > 0$ and $\sin (n-k)\theta < 0$.

Now, putting $z = \rho e^{i\theta}$ in the trinomial equation and separating real and imaginary parts, we find that $\rho^{n-k} \sin n\theta - \rho^n \sin(n-k)\theta - \sin k\theta = 0$. By Corollary 2.2, we have $\sin k\theta > 0$ for any feasible angle. Then, we can define the following function of $\rho$:

$$w(\rho) = -\rho^n \sin(n-k)\theta / \sin k\theta + \rho^{n-k} \sin n\theta / \sin k\theta - 1. \quad (1)$$

Thus, we prove the following important result.

**Proposition 2.3** For any feasible angle $\theta$ for the trinomial equation $z^n = \alpha z^k + (1 - \alpha)$ with $-\infty < \alpha < 0$, the function $w(\rho)$ of $\rho$ is increasing and vanishes for one and only one value of $\rho$ larger than or equal to 1.

**Proof.** Let us consider the trinomial equation in the case $-\infty < \alpha < 0$ and let $\theta$ be a feasible angle. Then, we have $\text{sgn}(\sin n\theta) = -\text{sgn}(\sin k\theta) = \text{sgn}(\sin(n-k)\theta)$. The function $w(\rho)$ given by (1) is differentiable with $w'(\rho) = \rho^{n-k-1} [(n-k) \sin n\theta - n\rho^k \sin(n-k)\theta] / \sin k\theta$.

On the other hand, let us divide the trinomial equation by $z^n$. When $\theta \neq j\pi/(n-k)$, $j \in \mathbb{N}$, one can have $\rho^k = (1-1/\alpha) \sin n\theta / \sin(n-k)\theta$. Hence, we conclude that
According to (2), we can deduce that \( w'(\rho) > 0 \). This means that \( w \) is an increasing function.

Moreover, one can find that \( w(1) \leq 0 \) and that \( w(\rho) \) tends to \(+\infty\) when \( \rho \) goes to \(+\infty\). It follows that \( w(\rho) \) vanishes for one and only one value of \( \rho \) larger than or equal to 1. Thus, the proof is achieved.

As consequence of Proposition 2.3, we state the next corollary.

**Corollary 2.4** The trajectories of roots of the trinomial equation \( z^n = \alpha z^k + (1 - \alpha) \) with \(-\infty < \alpha < 0\) are trinomial curves outside the unit disk.

The figure below sketches the family of trinomial curves \( Q(p, k, r, n) \) outside the unit disk.

![Trinomial curves](image)

### 3 Analogy between \( Q(p, k, r, n) \) and \( G(p, k, r, n) \)

Let us recall that any trinomial arc \( G(p, k, r, n) \) is the set of roots of the trinomial equation with \( 0 < \alpha < 1 \) and the feasible angles belong to the interval \([(2p + 1)\pi/k, 2\pi r/n]\), where \( n \) is an integer greater than or equal to 3, \( p \) and \( r \) are two integers verifying \( r \geq p + 1 \) and \( k \) is an integer such that \((2p + 1)n/2r < k < (2p + 1)n/(2r - 1)\).

In this trinomial equation with \( 0 < \alpha < 1 \), let us set \( x = 1/z \) and \( \sigma = \alpha/(\alpha - 1) \). If \( \rho' = |x| \), then \( \rho' = 1/\rho \). Then, we conclude that the situation \( 0 < \rho < 1 \) is equivalent to the situation \( 1 < \rho' < +\infty \) and that the case \( 0 < \alpha < 1 \) is equivalent to the case \(-\infty < \sigma < 0\).
Therefore, the trinomial equation $z^n = \alpha z^k + (1 - \alpha)$ where $0 < \alpha < 1$ and $\rho < 1$ becomes the equation $x^n = \sigma x^{n-k} + (1 - \sigma)$, where $-\infty < \sigma < 0$ and $\rho' > 1$. Moreover, the $\alpha$-free equation $\rho^n - k \sin n\theta - \rho^n \sin (n-k)\theta - \sin k\theta = 0$ becomes the following $\sigma$-free equation $(\rho')^n \sin n\theta - (\rho')^n \sin k\theta - \sin (n-k)\theta = 0$.

Hence, there is one basic remark: To pass from the case $0 < \alpha < 1$ to the case $-\infty < \alpha < 0$, it is sufficient to replace $k$ by $(n-k)$.

This remark is also valid for the feasible angles. In fact, an angle is feasible in the case $0 < \alpha < 1$ if $\text{sign}(\sin n\theta) = \text{sign}(\sin k\theta) = -\text{sign}(\sin (n-k)\theta)$ and it is feasible in the case $-\infty < \alpha < 0$ if $\text{sign}(\sin n\theta) = -\text{sign}(\sin (n-k)\theta) = \text{sign}(\sin k\theta)$.

Now, applying this basic remark to the trinomial curves $G(p, k, r, n)$ defined above, we will obtain trinomial curves solutions of the equation $z^n = \alpha z^k + (1 - \alpha)$ with $-\infty < \alpha < 0$ and for which the feasible angles belong to the interval $[(2p+1)\pi/(n-k), 2\pi r/n]$, where $p$ and $r$ are two integers such that $r \geq p + 1$ and $k$ is an integer satisfying $(2p+1)n/2r < (n-k) < (2p+1)n/(2r-1)$, i.e. $2(r-p-1)n/(2r-1) < k < [2(r-p)-1]n/2r$. This family of curves is exactly the family $Q(p, k, r, n)$ defined in the present work. Thus, this study allows us to affirm that an analogy exists between the trinomial curves $G(p, k, r, n)$ and the trinomial curves $Q(p, k, r, n)$.

References


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