Monotonicity of Trinomial Curves $H(p, k, r, n)$

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Abstract

In the present paper, our aim is to study the behavior of the trinomial curves $H(p, k, r, n)$. The monotonicity of this category of curves is completely established.

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1 Introduction

In the early 1900’s, trinomial equations were discussed in many papers and by a variety of authors. However, the first large discussion of these equations was established in 1980 by Fell [2]. She proved some results giving the form and the location of the trajectories of zeros of these equations, called trinomial curves. Nevertheless, she could not solve the monotonicity of these curves; problem which is pointed out in [2] and unsolved until now.

In the present note, we are interested in a category of trinomial curves denoted by $H(p, k, r, n)$. It will be shown that these curves are monotonic. From the following equation $\rho^{n-k} \sin n\theta - \rho^n \sin(n-k)\theta = \sin k\theta$ of trajectories of zeros of the trinomial equation, we will prove that the function $\rho = \rho(\theta)$ is decreasing for the totality of trinomial curves $H(p, k, r, n)$.

2 Basic definitions and properties

Consider the trinomial equation

$$z^n = \alpha z^k + (1 - \alpha),$$  \hspace{1cm} (1)
where $\alpha$ is a real number between 0 and 1, $n$ is an integer larger than one and $k = 1, 2, \ldots, n-1$. If $z = re^{i\theta}$, one has $\rho^n e^{in\theta} = \alpha \rho^k e^{ik\theta} + 1 - \alpha$. Separating real and imaginary parts, one gets

$$\rho^{n-k} \sin n\theta - \rho^n \sin (n-k)\theta = \sin k\theta.$$  \hspace{1cm} (2)

Because $\alpha$ is between 0 and 1, we are interested in the angles $\theta$ for which

$$\text{sign}(\sin n\theta) = \text{sign}(\sin k\theta) = -\text{sign}(\sin (n-k)\theta)$$  \hspace{1cm} (3)

Let us define a \textit{feasible angle} for the equation (1) with $0 < \alpha < 1$ as an angle $\theta$ which fulfills (3). Note that the cases $\alpha = 0$ and $\alpha = 1$ are two particular cases for the trinomial equation (1). When $\alpha = 0$, the zeros of (1) are the $n^{th}$ roots of unity and when $\alpha = 1$, the $n$ roots of (1) are exactly the $(n-k)^{th}$ roots of unity, which are simple roots and 0, a root of multiplicity $k$.

On the other hand, according to Lemma 3.2 of [1], all the trinomial curves corresponding to the case $0 < \alpha < 1$ are inside the unit disk. Moreover, since a symmetry exists between upper and lower half-planes, we can restrict the present study to the upper half-plane.

Equation (1) with $\alpha = 0$ possess as roots the $n^{th}$ roots of unity. When $\alpha$ moves from 0 to 1, the trajectories of the $n$ roots are continuous arcs inside the unit disk. Fell tells us in [2] that $k$ trajectories of these curves tend to zero such that its tangents are a line segments joining 0 and a $k^{th}$ root of $-1$. Also, she describes these $k$ trajectories and considers the sets: $C = \{n^{th}$ roots of unity\}, $D = \{(n-k)^{th}$ roots of unity\} and $E = \{k^{th}$ roots of $-1\}$. Let us restrict our attention to the case such that $\gamma \in C$ and $\delta$ be the unique nearest neighbor of $\gamma$ such that $\delta \in E$ and $\delta \notin D$. Fell shows in [2] that in this case, the feasible angles $\theta$ belong to intervals of length less than or equal to $\pi/n$ and bounded on one side by the argument of an $n^{th}$ root of unity and on the other side by the argument of a $k^{th}$ root of $-1$. In this work, we deal with these trinomial curves denoted by $H(p, k, r, n)$, for which the feasible angles $\theta$ belong to an interval of the form $[2\pi r/n, (2p + 1)\pi/k]$, where $p$ and $r$ are two integers. Indeed, $2\pi r/n$ is the argument of an $n^{th}$ root of unity and $(2p+1)\pi/k$ is the argument of a $k^{th}$ root of $-1$. Because the trajectories of roots of (1) with $0 < \alpha < 1$ for the case $n = 2$ are linear, this case is considered particular. Hence, we define the family of the trinomial curves $H(p, k, r, n)$ as follows.

Let $n$ be an integer larger than or equal to 5. The trinomial curve $H(p, k, r, n)$ is the set of roots of (1) such that $0 < \alpha < 1$ and the feasible angles belong to $[2\pi r/n, (2p + 1)\pi/k]$, where $p$ is an integer, $r$ is a nonzero integer such that $r \geq p+1$ and $k$ is an integer satisfying $(2p+1)n/(2r+1) < k < (2p+1)n/2r$. 

Proposition 2.1 Assume that $n$ is an integer larger than or equal to 5, $k$ is an integer such that $1 \leq k \leq n - 1$ and $0 < \alpha < 1$. In equation (1) with $(2p + 1)n/(2r + 1) < k < (2p + 1)n/2r$, where $p$ is an integer, $r$ is a nonzero integer verifying $r \geq p + 1$, any angle of $[2\pi r/n, (2p + 1)\pi/k]$ is feasible.

Proof. Suppose that $k$ verify $(2p + 1)n/(2r + 1) < k < (2p + 1)n/2r$. As $0 < k < n$, $p$ and $r$ satisfy $(2p + 1)/2r \leq 1$, i.e. $r \geq p + 1$. Let us consider an angle $\theta$ such that $2\pi r/n < \theta < (2p + 1)\pi/k$. One has $2\pi r < n\theta < (2p + 1)\pi n/k$. Because $(2p + 1)n/(2r + 1) < k$, it yields that $(2p + 1)\pi n/k < (2r + 1)\pi$ and that $\sin n\theta > 0$. Also, we find that $2\pi rk/n < k\theta < (2p + 1)\pi$. From the fact that $r \geq p + 1$ stems easily that $2\pi p < 2\pi rk/n$ and that $\sin k\theta > 0$. Moreover, we obtain that $2\pi r(1 - k/n) < (n - k)\theta < (2p + 1)\pi (n/k - 1)$. One gets $[2(r - p) - 1] \pi < 2\pi r(1 - k/n)$ and $(2p + 1)\pi (n/k - 1) < 2(r - p)\pi$. So $\sin(n - k)\theta < 0$. Fulfilling (3), $\theta$ is feasible. Thus, we achieve the proof.

Proposition 2.2 $\rho(\theta)$ is differentiable for each trinomial curve $H(p, k, r, n)$.

Proof. Assume that $H(p, k, r, n)$ is a trinomial curve. To show that $\rho(\theta)$ is differentiable means to show that the derivative $d\rho/d\theta$ exists and it’s well-defined. To begin, let us divide (1) by $z^n$. When $\theta \neq l\pi/(n - k)$, $l \in \mathbb{N}$, one has $\rho^k(\theta) = (1 - 1/\alpha) \sin n\theta / \sin(n - k)\theta$. From Proposition 2.1, the feasible angles $\theta$ satisfy $\sin n\theta > 0$ and $\sin(n - k)\theta < 0$. Let us put $l(\theta) = (1 - 1/\alpha) \sin n\theta / \sin(n - k)\theta$. As $0 < \alpha < 1$, the denominator of $l(\theta)$ is never zero and $l(\theta)$ is well-defined. Because $l$ is differentiable and positive, $\rho(\theta) = [l(\theta)]^{1/k}$ is also differentiable. So, $d\rho/d\theta$ exists and it’s well-defined.

3 Monotonicity of the curves $H(p, k, r, n)$

The main goal of this section is to show that the function $\rho(\theta)$ is monotonic for the trinomial curves $H(p, k, r, n)$. More precisely, the question is to prove that $d\rho/d\theta \neq 0$. Differentiating both sides of (2) with respect to $\theta$, we find

\[ [(n - k)\rho^{n-k-1} \sin n\theta - n \rho^{n-1} \sin(n - k)\theta] \frac{d\rho}{d\theta} = k \cos k\theta + (n - k) \rho^{n} \cos(n - k)\theta - n \rho^{n-k} \cos n\theta. \]

Supposing that $d\rho/d\theta = 0$, it yields that

\[
\begin{cases}
    k \cos k\theta + (n - k) \rho^{n} \cos(n - k)\theta - n \rho^{n-k} \cos n\theta = 0 \\
    \rho^{n-k} \sin n\theta - \rho^{n} \sin(n - k)\theta - \sin k\theta = 0.
\end{cases}
\]

The system above is equivalent to the following

\[
\begin{align*}
Z(\theta) \cdot \rho^{n-k} &= X(\theta) \\
Z(\theta) \cdot \rho^{n} &= Y(\theta)
\end{align*}
\]

(4)
with
\[ Z(\theta) = (n-k) \sin k\theta - k \cos n\theta \sin(n-k)\theta \\
X(\theta) = (n-k) \sin n\theta - n \sin(n-k)\theta \cos k\theta \\
Y(\theta) = (n-k) \sin k\theta \cos n\theta - k \sin(n-k)\theta. \]

By the system (4), we arrive at the relation
\[ Z(\theta) [\rho^n - \rho^{n-k}] = V(\theta) [1 - \cos k\theta] \quad (5) \]
with
\[ V(\theta) = -[n \sin(n-k)\theta + (n-k) \sin n\theta]. \]

To contradict the hypothesis \( d\rho/d\theta = 0 \), we need the remark below and the following propositions.

**Remark 3.1** Let \( H(p,k,r,n) \) be a trinomial curve. For all the feasible angles \( \theta \), we can distinguish the following two cases:

**Case 1):** \((2p+1)n/(2r+1) < k < 2(2p+1)n/(4r+1)\). We have \( \cos n\theta = 0 \) on \( \lfloor 2\pi r/n, (2p+1)\pi/k \rfloor \) if and only if \( \theta = (4r+1)\pi/2n \). In addition, we find that \( \cos n\theta > 0 \) for \( \theta < (4r+1)\pi/2n \) and \( \cos n\theta < 0 \) for \( \theta > (4r+1)\pi/2n \).

**Case 2):** \( 2(2p+1)n/(4r+1) \leq k < (2p+1)n/2r \). In this situation, \( \cos n\theta > 0 \) on the interval \( \lfloor 2\pi r/n, (2p+1)\pi/k \rfloor \).

**Proposition 3.2** Assume that \( H(p,k,r,n) \) is a trinomial curve. We have \( Y(\theta) > 0 \) for all the feasible angles.

**Proof.** Let \( \theta \) be a feasible angle in \( \lfloor 2\pi r/n, (2p+1)\pi/k \rfloor \). From the proof of Proposition 2.1, we obtain that \( \sin k\theta > 0 \) and \( \sin(n-k)\theta < 0 \). Thus, the sign of \( Y(\theta) \) depends on that of \( \cos n\theta \). Using Remark 3.1, we distinguish in Case 1) the two following subcases:

**Subcase 1.1):** \( \theta \) belongs to \( \lfloor 2\pi r/n, (4r+1)\pi/2n \rfloor \). In this case, \( \cos n\theta \geq 0 \). This implies that \( Y(\theta) > 0 \).

**Subcase 1.2):** \( \theta \) belongs to \( \lfloor (4r+1)\pi/2n, (2p+1)\pi/k \rfloor \). In the present situation, we have \( \cos n\theta < 0 \). Also, one gets \( (2r+1/2)\pi k/n < k\theta < (2p+1)\pi \).

Because \( r > p \), we find that \( (2p+1/2)/(2r+1/2) < (2p+1)/(2r+1) < k/n \). It yields that \( (2p+1/2)\pi < k\theta < (2p+1)\pi \) and that \( \cos k\theta < 0 \). Let us consider the function \( T(\theta) = Y(\theta)/\cos n\theta \cos k\theta = n \tan k\theta - k \tan n\theta \). \( Y(\theta) \) has the same sign as \( T(\theta) \). Observing that \( \tan n\theta < 0 \) and \( \tan k\theta < 0 \), the zeros of \( T'(\theta) = nk[\tan^2 k\theta - \tan^2 n\theta] \) are solutions of the equation \( \tan n\theta = k\theta \), which possess a unique root of the form \( \theta = j\pi/(n-k) \) where \( j \in \mathbb{N} \). But \( j\pi/(n-k) \in \lfloor (4r+1)\pi/2n, (2p+1)\pi/k \rfloor \) if and only if \( (2r+1/2)(1-k/n) < j < (2p+1)(n/k-1) \). Because \( k < 2(2p+1)n/(4r+1) \), so \( 2(r-p) - 1/2 < (2r+1/2)(1-k/n) \) and because \( (2p+1)n/(2r+1) < k \), so \( (2p+1)(n/k-1) < 2(r-p) \). Hence \( 2(r-p) - 1/2 < j < 2(r-p) \), which is not possible. We deduce that \( T(\theta) \) is a monotonic function. Because \( T(\theta) \) tends to
+∞ as θ tends on the right to (4r+1)π/2n and T((2p+1)π/k) > 0, we arrive at $T(\theta) > 0$ and that $Y(\theta) > 0$ for all the angles $\theta$ in $]4(r+1)\pi/2n, (2p+1)\pi/k[.$

On the other hand, in Case 2) of Remark 3.1, one has $Y(\theta) > 0$ for all the angles $\theta$ in $]2\pi r/n, (2p+1)\pi/k[.$ Thus, we achieve the proof.

**Proposition 3.3** Assume that $H(p, k, r, n)$ is a trinomial curve. The sign of $Z(\theta)$ changes according to the two cases:

Case 1): $(2p+1)n/(2r+1) < k < 2(2p+1)n/(4r+1).$ In this situation, there exists a feasible angle $\theta_0$, such that $Z(\theta_0) = 0.$ Moreover, $Z(\theta) > 0$ for $\theta < \theta_0$ and $Z(\theta) < 0$ for $\theta > \theta_0.$

Case 2): $2(2p+1)n/(4r+1) \leq k < (2p+1)n/2r.$ In this case, $Z(\theta) > 0$ for all the feasible angles.

**Proof.** Let $H(p, k, r, n)$ be a trinomial curve and $\theta$ be a feasible angle. Let us recall that $Z(\theta) = (n-k)\sin k\theta - k\cos n\theta \sin(n-k)\theta.$ We know that $\sin k\theta > 0$ and $\sin(n-k)\theta < 0.$ Applying the remark above, we distinguish in Case 1) the two subcases below:

Subcase 1.1): $\theta$ belongs to $]2\pi r/n, (4r+1)\pi/2n[.$ So, we have $\cos n\theta \geq 0.$ It follows that $Z(\theta) > 0.$

Subcase 1.2): $\theta$ belongs to $](4r+1)\pi/2n, (2p+1)\pi/k[.$ We find that $\cos n\theta < 0.$ Moreover, because $(2p+1)n/(2r+1) < k < 2(2p+1)n/(4r+1),$ we can easily prove that $\cos(n-k)\theta < 0.$ Now, let us consider the function $W(\theta) = Z(\theta)/\cos n\theta \cos(n-k)\theta = (n-k)\tan n\theta - n\tan(n-k)\theta.$ The sign of $Z(\theta)$ is opposed to that of $W(\theta).$ Remarking that $\tan n\theta < 0$ and $\tan(n-k)\theta < 0,$ the zeros of $W'(\theta)$ are those of the equation $\tan n\theta = \tan(n-k)\theta,$ which possess a unique root of the form $\theta = j\pi/k$ where $j \in \mathbb{N}.$ However, $j\pi/k$ can not belong to the interval $](4r+1)\pi/2n, (2p+1)\pi/k[.$ because $j$ is an integer. This implies that $W(\theta)$ is a monotonic function. Since $W(\theta)$ tends to $-\infty$ as $\theta$ tends on the right to $(4r+1)\pi/2n$ and $W((2p+1)\pi/k) > 0,$ there exists an angle $\theta_0$ in $](4r+1)\pi/2n, (2p+1)\pi/k[.$ such that $W(\theta_0) = 0$ and that $Z(\theta_0) = 0.$ Moreover, $Z(\theta) > 0$ for $\theta < \theta_0$ and $Z(\theta) < 0$ for $\theta > \theta_0.$

Therefore, from the study above, we assert that in Case 1), there exists a feasible angle $\theta_0$ such that $Z(\theta_0) = 0.$ Moreover, $Z(\theta) > 0$ for $\theta < \theta_0$ and $Z(\theta) < 0$ for $\theta > \theta_0.$

To achieve the proof, let us apply Remark 3.1 an other time. In Case 2), we deduce that $Z(\theta) > 0$ for all the angles $\theta$ in $]2\pi r/n, (2p+1)\pi/k[.$

**Proposition 3.4** For any trinomial curve $H(p, k, r, n),$ we have $V(\theta) > 0$ for all the feasible angles.

**Proof.** Let $H(p, k, r, n)$ be a trinomial curve and let $\theta$ be a feasible angle. To start, let us show that the function $V(\theta)$ is monotonic. Supposing that $V'(\theta) = -n(n-k)[\cos(n-k)\theta + \cos n\theta] = 0,$ we obtain two possibilities:
\[ \theta = (2j-1)\pi/k \text{ or } \theta = (2j+1)\pi/(2n-k), \ j \in \mathbb{N}. \] As for the first situation, it’s impossible because the angle \((2j-1)\pi/k\) isn’t feasible. Supposing that this angle is feasible, we arrive at the double inequality \(rk/n + 1/2 < j < p + 1\). As \((2p + 1)n/(2r + 1) < k\) and \(r > p\), so \(p + 1/2 < rk/n + 1/2\). This last inequality contradicts the fact that \(j \in \mathbb{N}\). On the other hand, also the angle \((2j + 1)\pi/(2n - k)\) is not feasible. In fact, assuming that it is the case, we find that \(2r(1-k/2n) - 1/2 < j < (2p + 1)(n/k - 1/2) - 1/2\). Because \(k < (2p + 1)n/2r\), so \(2r - p - 1 < 2r(1-k/2n) - 1/2\) and because \((2p + 1)n/(2r + 1) < k\), so \((2p + 1)(n/k - 1/2) - 1/2 < 2r - p\). It follows that \(2r - p - 1 < j < 2r - p\), which is not possible. Therefore, \(V(\theta)\) is a monotonic function on \([2\pi r/n, (2p + 1)\pi/k]\). In addition, as \(V(2\pi r/n) > 0\) and \(V((2p + 1)\pi/k) > 0\), we deduce that \(V(\theta) > 0\) for any feasible angle \(\theta\).

Now, we are able to state the following main result.

**Theorem 3.5** Let \(H(p,k,r,n)\) be a trinomial curve. For all the feasible angles \(\theta\), the function \(\rho(\theta)\) is decreasing.

**Proof.** Let \(H(p,k,r,n)\) be a trinomial curve and let \(\theta\) be a feasible angle. By Proposition 3.2, one has \(Y(\theta) > 0\) and from Proposition 3.4, one gets \(V(\theta) > 0\). However, Proposition 3.3 tells us that the sign of \(Z(\theta)\) changes according to two cases. In Case 1), there exists a feasible angle \(\theta_0\), such that \(Z(\theta_0) = 0\). So, we distinguish two subcases:

**Subcase 1.1:** \(\theta\) belongs to \([2\pi r/n, \theta_0]\). In this situation, \(Z(\theta) > 0\). Considering the relation \(Z(\theta)[\rho^n - \rho^{n-k}] = V(\theta)[1 - \cos k\theta]\) given by (5), it yields that \(\rho^n > \rho^{n-k}\). This provides a contradiction because \(\rho < 1\).

**Subcase 1.2:** \(\theta\) belongs to \([\theta_0, (2p + 1)\pi/k]\). We find that \(Z(\theta) \leq 0\). This contradicts the equation \(Z(\theta) \cdot \rho^n = Y(\theta)\) of (4) and the fact that \(\rho > 0\).

On the other side, in Case 2) of Proposition 3.3, we have \(Z(\theta) > 0\) on the interval \([2\pi r/n, (2p + 1)\pi/k]\). As in Subcase 1.1, it is not possible.

Consequently, the hypothesis \(d\rho/d\theta = 0\) is impossible. Thus, \(\rho(\theta)\) is monotonic on the interval of the feasible angles. Now, let us put \(\theta = 2\pi r/n\) in the equation (2). We obtain that \((\rho^n - 1)\sin(2\pi r k/n) = 0\). Because \(2\pi p < 2\pi r k/n < (2p + 1)\pi\), one gets \(\sin(2\pi r k/n) \neq 0\). It follows that \(\rho(2\pi r/n) = 1\). Moreover, the trinomial curves \(H(p,k,r,n)\) are inside the unit disk, i.e. \(\rho(\theta)\) varies between 0 and 1. Hence, \(\rho(\theta)\) is a decreasing function.

**References**


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