Common Fixed Points of Two Quasi-Contractive Operators in Normed Spaces by Iteration

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Abstract. We prove a theorem to approximate common fixed points of two quasi-contractive operators on a normed space through an iteration process with errors and more general than the Ishikawa iteration process. Our result generalizes and improves upon, among others, the corresponding result of Berinde [1] in the following two different directions: (i) more general iteration process with error terms (ii) wider class of mappings.

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1. Introduction and Preliminaries

Throughout this paper, \( \mathbb{N} \) will denote the set of all positive integers. Let \( C \) be a nonempty convex subset of a normed space \( E \) and \( T : C \to C \) be a mapping. Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences in \([0,1]\).

The Mann iteration process is defined by the sequence \( \{x_n\} \) (see [12]):

\[
\begin{align*}
\{x_n\} & \quad \{ \begin{array}{l}
    x_1 = x \in C, \\
    x_{n+1} = (1 - a_n)x_n + a_nTx_n, \quad n \in \mathbb{N}.
\end{array}
\end{align*}
\]

The sequence \( \{x_n\} \) defined by

\[
\begin{align*}
\{x_n\} & \quad \{ \begin{array}{l}
    x_1 = x \in C, \\
    x_{n+1} = (1 - a_n)x_n + a_nTy_n, \\
    y_n = (1 - b_n)x_n + b_nTx_n, \quad n \in \mathbb{N}
\end{array}
\end{align*}
\]

is known as the Ishikawa iteration process [5].

Liu [11] introduced the concept of Ishikawa iteration process with errors by the sequence \( \{x_n\} \) defined as follows:
\begin{align}
  x_1 &= x \in C, \\
  x_{n+1} &= (1 - a_n) y_n + a_n T y_n + u_n, \\
  y_n &= (1 - b_n) x_n + b_n T x_n + v_n, \quad n \in \mathbb{N}
\end{align}

where \( \{a_n\} \) and \( \{b_n\} \) are sequences in \([0, 1]\) and \( \{u_n\} \) and \( \{v_n\} \) satisfy \( \sum_{n=1}^{\infty} \|u_n\| < \infty, \quad \sum_{n=1}^{\infty} \|v_n\| < \infty \). This surely contains both (1.1) and (1.2).

In 1998, Xu [19] introduced more satisfactory error terms in the sequence defined by:

\begin{align}
  x_1 &= x \in C, \\
  x_{n+1} &= a_n T y_n + b_n x_n + c_n u_n, \\
  y_n &= a'_n T x_n + b'_n x_n + c'_n v_n, \quad n \in \mathbb{N}
\end{align}

where \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\} \) are sequences in \([0, 1]\) and \( \{u_n\}, \{v_n\} \) are bounded sequences in \( C \). This again contains both (1.1) and (1.2) as special cases.

For two self mappings \( S \) and \( T \) of \( C \), the Mann and Ishikawa iteration processes have been generalized by Das and Debata [3] as follows:

\begin{align}
  x_1 &= x \in C, \\
  x_{n+1} &= (1 - a_n) x_n + a_n S y_n \\
  y_n &= y_{n-1} + (1 - b_n) x_n + b_n T x_n, \quad n \in \mathbb{N}
\end{align}

They used this iteration process to find common fixed points of quasi-nonexpansive mappings in a uniformly convex Banach space. Takahashi and Tamura [18] studied it for the case of two nonexpansive mappings under different conditions in a strictly convex Banach space. For the case of two asymptotically nonexpansive mappings, we refer to Khan and Takahashi [10].

Fukhar and Khan [4] have approximated the common fixed points of two asymptotically nonexpansive mappings using the iteration process with errors in the sense of Liu [11]:

\begin{align}
  x_1 &= x \in C, \\
  x_{n+1} &= (1 - a_n) y_n + a_n S^n y_n + u_n, \\
  y_n &= (1 - b_n) x_n + b_n T^n x_n + v_n, \quad n \in \mathbb{N}
\end{align}

Very recently, Khan and Fukhar [9] studied the iteration process for two nonexpansive mappings using errors in the sense of Xu [19]:

\begin{align}
  x_1 &= x \in C, \\
  x_{n+1} &= a_n S y_n + b_n x_n + c_n u_n, \\
  y_n &= a'_n T x_n + b'_n x_n + c'_n v_n, \quad n \in \mathbb{N}
\end{align}
where \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\} \) are sequences in \([0, 1]\) such that \( a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n \) and \( \{u_n\}, \{v_n\} \) are bounded sequences in \( C \). Clearly, this iteration process contains all the processes (1.1), (1.2), (1.4) and (1.5) as its special cases.

It is remarked that approximating common fixed points of two mappings has its own importance as it has a direct link with the minimization problem (see, for example, [17]).

We recall some definitions in a metric space \((X, d)\). A mapping \( T : X \to X \) is called an \( a \)-contraction if
\[
(1.8) \quad d(Tx, Ty) \leq ad(x, y) \text{ for all } x, y \in X,
\]
where \( 0 < a < 1 \).

The map \( T \) is called Kannan mapping [6] if there exists \( b \in (0, \frac{1}{2}) \) such that
\[
(1.9) \quad d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X.
\]

A similar definition is due to Chatterjea [2]: there exists \( c \in (0, \frac{1}{2}) \) such that
\[
(1.10) \quad d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X.
\]

Combining the above three definitions, Zamfirescu [20] proved the following important result.

**Theorem 1.** Let \((X, d)\) be a complete metric space and \( T : X \to X \) a mapping for which there exist real numbers \( a, b \) and \( c \) satisfying \( 0 < a < 1 \), \( b \in (0, \frac{1}{2}) \), \( c \in (0, \frac{1}{2}) \) such that for each pair \( x, y \in X \), at least one of the following conditions holds:
\[
(\z_1) \quad d(Tx, Ty) \leq ad(x, y) \text{ for all } x, y \in X
\]
\[
(\z_2) \quad d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X
\]
\[
(\z_3) \quad d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X.
\]

Then \( T \) has a unique fixed point \( p \) and the Picard iteration \( \{x_n\} \) defined by
\[
x_{n+1} = Tx_n, \quad n \in \mathbb{N}
\]
converges to \( p \) for any arbitrary but fixed \( x_1 \in X \).

An operator \( T \) satisfying the contractive conditions \((\z_1), (\z_2)\) and \((\z_3)\) in the above theorem is called Zamfirescu operator.

In 2005, Berinde [1] introduced a new class of operators on a normed space \( X \) satisfying
\[
(1.11) \quad \|Tx - Ty\| \leq \delta \|x - y\| + L \|Tx - x\|
\]
for any \( x, y \in X \), \( 0 < \delta < 1 \) and \( L \geq 0 \).

He proved that this class is wider than the class of Zamfirescu operators and used the Ishikawa iteration process to approximate fixed points of this class of operators in a normed space.

We now generalize the class introduced by Berinde [1] to the case of two self mappings \( S, T \) of a convex set \( C \). In our case, \( S \) and \( T \) satisfy
\[\parallel T x - T y \parallel \leq \delta \parallel x - y \parallel + L \parallel T x - x \parallel \]  

(1.12)

and

\[\parallel S x - S y \parallel \leq \delta \parallel x - y \parallel + L \parallel S x - x \parallel \]  

(1.13)

hold for \(0 < \delta < 1\) and \(L \geq 0\). Clearly (1.11) can be retrieved from (1.13) when \(S = T\).

Thus the class of maps satisfying (1.12) and (1.13) is generally larger than the one determined by (1.11) and, in turn, wider than the class of Zamfirescu operators.

In this paper, we approximate the common fixed points of two mappings satisfying (1.12) and (1.13) by using the iteration process (1.7).

We shall need the following lemma.

\textbf{Lemma 1.} [11] Let \(\{r_n\}, \{s_n\}, \{t_n\}\) and \(\{k_n\}\) be sequences of nonnegative numbers satisfying

\[r_{n+1} \leq (1 - s_n)r_n + s_nt_n + k_n \quad \text{for all } n \geq 1.\]

If \(\sum_{n=1}^{\infty} s_n = \infty\), \(\lim_{n \to \infty} t_n = 0\) and \(\sum_{n=1}^{\infty} k_n < \infty\) hold, then \(\lim_{n \to \infty} r_n = 0\).

In the sequel, the set of common fixed points of the maps \(S\) and \(T\) will be denoted by \(F\).

\section{Main Result}

Usually, iterative results are obtained through a fixed point theorem. Following Berinde [1], we obtain such a result without employing any fixed point theorem. The use of Lemma 1 in our proof will further simplify the methods of proof of Theorem 1 in [1].

We now prove our main theorem as follows.

\textbf{Theorem 2.} Let \(C\) be a nonempty closed bounded convex subset of a normed space \(E\). Let \(S,T : C \to C\) be two operators satisfying (1.12) and (1.13). Let \(\{x_n\}\) be defined by the iterative process (1.7). If \(F \neq \phi\), \(\sum_{n=1}^{\infty} a_n = \infty\), \(\sum_{n=1}^{\infty} c_n < \infty\) and \(\lim_{n \to \infty} c_n' = 0\), then \(\{x_n\}\) converges strongly to a common fixed point of \(S\) and \(T\).

\textbf{Proof.} Assume that \(F \neq \phi\). Let \(w \in F\). Then

\[
\parallel x_{n+1} - w \parallel = \parallel a_nSy_n + b_n x_n + c_n u_n - (a_n + b_n + c_n)w \parallel
\]

\[
= \parallel a_n(Sy_n - w) + (1 - a_n)(x_n - w) + c_n(u_n - x_n) \parallel
\]

\[\leq a_n \parallel Sy_n - w \parallel + (1 - a_n) \parallel x_n - w \parallel + c_n \parallel u_n - x_n \parallel.
\]

(2.1)

Now for \(x = w\) and \(y = y_n\), (1.13) gives

\[
\parallel Sy_n - w \parallel \leq \delta \parallel y_n - w \parallel
\]

(2.2)
Also,

\[\|y_n - w\| \leq a'_n \|Tx_n - w\| + (1 - a'_n) \|x_n - w\| + c'_n \|v_n - x_n\|.\]

Making use of (1.12) with \(x = w\) and \(y = x_n\), we get

\[\|Tx_n - w\| \leq \delta \|x_n - w\|.\]

Since \(C\) is bounded, therefore we can choose a number \(M\) such that \(M \geq \max(\sup_{n \geq 1} \|u_n - x_n\|, \sup_{n \geq 1} \|v_n - x_n\|)\). Then using (2.2), (2.3) and (2.4) in (2.1), we obtain

\[
\|x_{n+1} - w\| \\ \leq a_n \delta [a'_n \|x_n - w\| + (1 - a'_n) \|x_n - w\| + c'_n M] \\ + (1 - a_n) \|x_n - w\| + c_n M \\ = [a_n a'_n \delta^2 + a_n \delta (1 - a'_n) + (1 - a_n)] \|x_n - w\| \\ + a_n \delta c'_n M + c_n M \\ = [1 - (a_n a'_n \delta + a_n) (1 - \delta)] \|x_n - w\| + a_n \delta c'_n M + c_n M \\ \leq [1 - (1 - \delta)^2 a_n] \|x_n - w\| + a_n \delta c'_n M + c_n M
\]

By Lemma 1, with \(r_n = \|x_n - w\|, s_n = (1 - \delta)^2 a_n, t_n = \frac{\delta c'_n}{(1 - \delta)\pi} M\) and \(k_n = c_n M\), we get that \(\lim_{n \to \infty} \|x_n - w\| = 0\). Consequently \(x_n \to w \in F\) and this completes the proof.

Our next theorem is proved for the scheme (1.5). This important result is actually a corollary of Theorem 2 above and generalises a number of results as seen below.

**Theorem 3.** Let \(C\) be a nonempty closed convex subset of a normed space \(E\). Let \(S, T : C \to C\) be two operators satisfying (1.12) and (1.13). Let \(\{x_n\}\) be defined by the iterative process (1.5). If \(F \neq \emptyset\) and \(\sum_{n=1}^{\infty} a_n = \infty\), then \(\{x_n\}\) converges strongly to a common fixed point of \(S\) and \(T\).

**Proof.** Choosing \(c_n = 0 = c'_n\) in (1.7) and making some obvious changes in the proof of Theorem 2, we get the desired result.

If we take \(S = T\) in Theorem 3, we obtain:

**Corollary 1.** ([1], Theorem 1) Let \(C\) be a nonempty closed convex subset of a normed space \(E\). Let \(T : C \to C\) be an operator satisfying (1.11). Let \(\{x_n\}\) be defined through the iterative process (1.2). If \(F(T) \neq \emptyset\) and \(\sum_{n=1}^{\infty} a_n = \infty\), then \(\{x_n\}\) converges strongly to the unique fixed point of \(T\).

**Corollary 2.** Let \(E\) be a Banach space and \(C\) a nonempty closed convex subset of \(E\). Let \(T : C \to C\) be a Zamfirescu operator. Let \(\{x_n\}\) be defined by the Ishikawa iteration process (1.2) with \(\sum_{n=1}^{\infty} a_n = \infty\). Then \(\{x_n\}\) converges strongly to the unique fixed point of \(T\).

**Proof.** The operator \(T\) has a unique fixed point by Theorem 1 and hence the result follows from Corollary 1.
Corollary 3. Let $E$ be a Banach space and $C$ a nonempty closed convex subset of $E$. Let $T : C \to C$ be a Zamfirescu operator. Let $\{x_n\}$ be defined by the Mann iteration process (1.1) with $\sum_{n=1}^{\infty} a_n = \infty$. Then $\{x_n\}$ converges strongly to the unique fixed point of $T$.

Proof. Set $b_n = 0$ for all $n \in \mathbb{N}$, in Corollary 2. ■

Remark. (1) The contractive condition (1.8) makes $T$ a continuous function on $X$ while this is not the case with the contractive conditions (1.9) – (1.12).

(2) The Chatterjea’s and the Kannan’s contractive conditions (1.9) and (1.10) are both included in the class of Zamfirescu operators and so their convergence theorems for the Ishikawa iteration process are obtained in Corollary 1. In particular, Theorem 2 and Corollary due to Kannan [8] are obtained on an unbounded domain for the Ishikawa iteration process. Theorems 2 and 3 of Kannan [6] and Theorem 3 of Kannan [7] are also special cases of Corollary 1.

(3) Theorem 4 of Rhoades [15] in the context of Mann iteration on a uniformly convex Banach space has been extended in Corollary 2 to the case of an Ishikawa iteration on arbitrary Banach space.

(4) In Corollary 2, Theorem 8 of Rhoades [16] is generalized to the setting of an arbitrary Banach space.

(5) Our result also generalizes Theorem 5 of Osilike [13] and Theorem 2 of Osilike [14].

References


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