A New Approach to Stability of

Impulsive Differential Equations

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Abstract

In this work, new approach to stability theory of impulsive differential equations is proposed. Instead of putting all components of the state variable \( x \) in one Liapunov function, several functions of partial components of \( x \), which can be much easier constructed, are used so that the conditions ensuring that stability are simpler and less restrictive. Also, an example is given to illustrate the advantages of the obtained results.

Keywords: Impulsive differential equations; Uniform Stability; uniform Asymptotic Stability; Liapunov Functions

Introduction

It is known that the method of liapunov –Razumikhin functions has been very powerful and effective in the study of impulsive differential equations [2-4]. To our knowledge one always puts all components of the state variable \( x \) together into one function \( V(t,x) \), and impose certain conditions on \( V(t,x) \), \( D^+V(t,x) \) and \( V(t_k,x+I_k(x)) \) (where \( t_k \) is impulsive point) to guarantee the required stability. Unfortunately, to construct such functions is rather difficult. As a matter of fact, with few exceptions (to the best of our knowledge) all the known examples to illustrate the methods of Liapunov-Razumikhin functions are of scalar equations. This show the disadvantages of using one function involving all components of \( x \), which restrict the applications of Liapunov-Razumikhin function method to impulsive differential systems.
Inspired by the idea in [5] dealing with differential equations, we develop a new technique in studying the stability of impulsive differential equation in which components $x_1, x_2, x_3, \ldots, x_n$ of $x$ are divided into two groups; correspondingly, two functions $V_1(t, x_1, \ldots, x_m)$ and $V_2(t, x_{m+1}, \ldots, x_n)$ are adopted, then according to the case of $V_1 \geq V_2 \text{or} V_1 \leq V_2$ and $V_1(t_k) \geq V_2(t_k) \text{or} V_1(t_k) \leq V_2(t_k)$ the certain conditions on $D^+ V_1 \text{or} D^+ V_2$ and $V_1(t_k) \text{or} V_2(t_k)$ are imposed to guarantee the required stability. In this way, to construct the suitable functions is rather easy, and the obtained conditions are less restrictive. Furthermore, it will be shown that the components of $x$ can be actually divided into more groups and correspondingly, more functions can be adopted. Therefore, this technique is rather flexible.

Preliminaries

We consider the impulsive differential equations

$$
\begin{align*}
x' &= f(t, x), t \geq t_0, t \neq t_k \\
x(t_k^+) &= x(t_k^-) + I_k(x(t_k)), k \in Z^+ 
\end{align*}
$$

Where $Z^+$ is the set of all positive integers. $f \in ([t_0, \infty) \times \Omega, R^n), \Omega$ is an open subset of the $n$-dimensional Euclidean space with an arbitrary norm $I_k \in C(R^n, R^n)$ for $k \in Z^+$

$$
t_0 \leq t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots \text{with } t_k \to \infty \text{ as } k \to \infty
$$

$x'(t)$ denotes the right-hand derivative of $x(t)$ at each point of discontinuities.

For given $t \geq t_0$ and $x_0 \in \Omega$, the initial value problem of equation (1) is

$$
\begin{align*}
x' &= f(t, x), t \geq t_0, t \neq t_k \\
x(t_k^+) &= x(t_k^-) + I_k(x(t_k)), k \in Z^+ \\
x(t_0) &= x_0 
\end{align*}
$$

Lemma1:-(Ballinger and Lui, 2000[1]), Assume that

(i) $f(t, x)$ is a measurable function of $t$ for all $t \in R^+$ and for all $x \in \Omega$

(ii) There exist $g, h \in L^\infty(R^+, R^+)$, such that $\|f(t, x_0)\| \leq g(t) + h(t)\|x_0\|

for all $x_0 \in \Omega$

(iii) For each compact set $F \subset R^n$, there exists $\Gamma \in L^\infty(R^+, R^+)$ such that
\[ \|f(t,x_1) - f(t,x_2)\| \leq \Gamma(t) \|x_1 - x_2\| \text{ for all } t \in R^+ \text{ and for all } x_1, x_2 \in F \]

Then for each \( t \geq t_0 \) and \( x \in \Omega \), there exists a unique solution \( x(t) \) of (2) and it exists.
For all \( t \in [0, \infty) \)

Throughout this paper we assume \( f(t,0) = 0 \) and \( I_k(0) = 0 \) so that \( x(t) = 0 \) is a solution of equation (1) which is called the zero solution.

The function \( V : [0, \infty) \times R^n \to R^+ \) belongs to \( V^m(.) \) if
(1) The function \( V \) is continuous on \([t_{k-1}, t_k) \times R^m, k \in Z^+ \) and for all \( t \geq t_0 \),
\( V(t,0) = 0 \)
(2) \( V(t,x) \) is locally lipschitzian in \( x \in R^m \).
(3) For each \( k=1,2,3, \ldots \) the following limits exists finitely:
\[
\lim_{x \to 0} V(t, y) = v(t_k, x) \\
(t, y) \to (t_k, x)
\]
Let \( V \in V^m(.) \), for \( (t,x) \in [t_{k-1}, t_k) \times R^m \), \( k=1,2,3, \ldots \), \( D^+V \) is defined as
\[
D^+V(t,x(t)) = \lim_{\delta \to 0} \frac{1}{\delta} [V(t + \delta), x(t + \delta)) - V(t,x)]
\]
Where \( x(t) \) is the solution of equation (1).
For any \( \delta > 0 \), Let \( S(\delta) = \{ x \in \Omega : \|x\| < \delta \} \)

Definition 1:- The zero solution of equation (1) is said to be
(1) Uniformly stable, if for any \( t \geq t_0 \) and \( \epsilon > 0 \), there exists a \( \delta = \delta(\epsilon) > 0 \) such that \( x_0 \in S(\delta) \) implies \( \|x(t, t_0, x_0)\| < \epsilon \)
(2) Uniformly asymptotically stable, if it is uniformly stable and there exists a \( \delta > 0 \) such that for any \( \epsilon > 0 \) there is a \( T = T(\epsilon) > 0 \) such that \( t \geq t_0 \) and \( x_0 \in S(\delta) \) together imply \( \|x(t, t_0, x_0)\| < \epsilon \) for \( t \geq t_0 + T \).

Definition 2:- Let \( K = \{ a(u) \in C[R^+, R^+] \) , increasing , \( a(0)=0 \} \)
\( D = \{ W \in C[R^+, R^+] \) , \( W(0) = 0, W(s) > 0, s > 0 \} \)
In what follows we will split \( x = (x_1, x_2, \ldots, x_n) \) into several vectors, say,
\( (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}), (x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}), \ldots, (x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}) \),
and
\[ x = (x_1^{(1)}, \ldots, x_n^{(1)}, x_1^{(2)}, \ldots, x_n^{(2)}, \ldots, x_1^{(m)}, \ldots, x_n^{(m)}), \]
\[ \sum_{j=1}^{m} x_j = n \]

For the sake of convenience, we denote
\[ x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)}), j = 1, 2, 3, \ldots, m, \text{and} \]
\[ x = (x^{(1)}, x^{(2)}, \ldots, x^{(m)}) \]

Note that the order of components in \( x^{(j)} \) is not necessarily the same as in \( x \).

Let \[ |x^{(j)}| = \max \{ |x_k^{(j)}| : 1 \leq k \leq nj \} \]
and
\[ |x| = \max \{ |x^{(j)}| : 1 \leq j \leq m \} \]

**MAIN RESULT**

For the sake of the simplicity, we start with the case of \( m=2 \) and first establish the following results on uniform stability.

**Theorem 1.** Suppose there exist \( V_1(t, x^{(1)}) \in V^{n_1}(.) \) and \( V_2(t, x^{(2)}) \in V^{n_2}(.) \), \( n = n_1 + n_2 \) such that

(i) \[ a_j(\{ |x^{(j)}| \}) \leq V_j(t, x^{(j)}) \leq b_j(\{ |x^{(j)}| \}) , j = 1, 2 \]

(ii) When \( V_1(t, x^{(1)}(t)) \geq V_2(t, x^{(2)}(t)) \), there holds \( D^+ V_j(t, x^{(j)}(t)) \leq 0 \)

When \( V_1 \left( t, x^{(1)}(t) \right) \leq V_2 \left( t, x^{(2)}(t) \right) \), there holds \( D^+ V_2(t, x^{(2)}(t)) \leq 0 \)

(iii) For all \( K \in Z^+ \), when \( V_1(t_k^-, x^{(1)}(t_k^-)) \geq V_2(t_k^-, x^{(2)}(t_k^-)) \) there holds
\[ \text{Max} \{ V_1(t_k^- x^{(1)}(t_k^-)), V_2(t_k^-, x^{(2)}(t_k^-)) \} \leq (1 + d_k) V_1(t_k^-, x^{(1)}(t_k^-)) \]

When \( V_1(t_k^-, x^{(1)}(t_k^-)) \leq V_2(t_k^-, x^{(2)}(t_k^-)) \) there holds
\[ \text{Max} \{ V_1(t_k^- x^{(1)}(t_k^-)), V_2(t_k^-, x^{(2)}(t_k^-)) \} \leq (1 + d_k) V_2(t_k^-, x^{(1)}(t_k^-)) \]

Where \( d_k \geq 0 \), \( \sum_{k=1}^{\infty} d_k < \infty \), \( a_j, b_j \in K \).
\[ j=1,2, \, x(t)=(x^{(1)}(t), x^{(2)}(t)) \] is the solution of (1). Then the zero solution of (1) is uniformly stable.

**Proof.** Let \( x(t) = x(t, t_0, x_0) \) is the solution of (1). Define a function \( V(t) \) as follows.

\[
V(t) = \begin{cases} 
V_1 \left( t, x^{(1)}(t) \right), & V_1 \left( t, x^{(1)}(t) \right) \geq V_2 \left( t, x^{(1)}(t) \right) \\
V_2 \left( t, x^{(1)}(t) \right), & V_2 \left( t, x^{(1)}(t) \right) \geq V_1 \left( t, x^{(1)}(t) \right)
\end{cases}
\]

For short, we denote, from now on
\[
V_1(t) = V_1 \left( t, x^{(1)}(t) \right) \\
V_2(t) = V_1 \left( t, x^{(1)}(t) \right)
\]

We point out first that for any \( t \geq t_0 \)

\[
\left[ a_1 \left( |x^{(1)}(t)| \right) + a_2 \left( |x^{(2)}(t)| \right) \right] / 2 \leq V(t) \leq b_1 \left( |x^{(1)}(t)| \right) + b_2 \left( |x^{(2)}(t)| \right)
\]

In fact, if \( V_1(t) \geq V_2(t) \), then
\[
V(t) = V_1(t) \geq [V_1(t) + V_2(t)] / 2 \geq \left[ a_1 \left( |x^{(1)}(t)| \right) + a_2 \left( |x^{(2)}(t)| \right) \right] / 2
\]

If \( V_1(t) \leq V_2(t) \), then
\[
V(t) = V_2(t) \geq [V_1(t) + V_2(t)] / 2 \geq \left[ a_1 \left( |x^{(1)}(t)| \right) + a_2 \left( |x^{(2)}(t)| \right) \right] / 2
\]

Obviously, \( V(t) \leq V_1(t) + V_2(t) \leq b_1 \left( |x^{(1)}(t)| \right) + b_2 \left( |x^{(2)}(t)| \right) \)

Next we claim that for \( t \geq t_0 \)
\[
D^+ V(t) \leq 0
\]

\[
V(t_k) \leq (1 + d_k) V(t^-_k), \, K \in Z^+ \quad \text{(5)}
\]

In fact, suppose \( V_1(t_0) \geq V_2(t_0) \) and there exists \( r_1 > t_0, \)
\[
V_1(t) \geq V_2(t), \quad t \in [t_0, r_1]
\]

By (3), we get \( V(t) = V_1(t) \) for all \( t \in [t_0, r_1] \)

Case-I: If \( t = t_j \) for some \( j \in Z^+ \), then by (iii)
\[
V(t_k) = V_1(t_k) \leq (1 + d_k) V(t^-_k) = (1 + d_k) V(t^-_k)
\]
Case-II : t is not a time of impulse effect .
If $V_1(t) \geq V_2(t)$ then
$V(t) = V_1(t)$
So by (ii) we have arrive at assertion that (5) is true for all $t \geq t_0$.
Otherwise there exists a $r_2>r_1$ such that $V_1(t) \leq V_2(t)$ when
$r_1 = t_i$ , for some $i \in Z^+$ we have $V_1(t_i^-) \geq V_2(t_i^-)$ and
$V_1(t) \leq V_2(t)$ .
In this case , by (ii) we have
$V(t_i) \leq (1 + d_i) V_1(t_i^-) = (1 + d_i) V(t_i^-)$
When $r_1 \neq t_i$ , we set $V(t) = V_2(t)$ for all $t \in [r_1, r_2)$

Case-I: If $t = t_j$ for some $j \in Z^+$ then by (iii)
$V(t_k) = V_2(t_k) \leq (1 + d_k) V_2(t_k^-) = (1 + d_k) V(t_k^-)$

Case-II: t is not a time of impulse effect .
If $V_1(t) \geq V_2(t)$ then
$V(t) = V_1(t)$
So by (ii) we have $D^+V(t) = D^+V_1(t) \leq 0$

If $r_2 = \infty$ then by (5) holds for all $t \geq \sigma$. Otherwise , repeat the above argument to arrive at the assertion that (5) is valid for all $t \geq t_0$.
As for the case of $V_1(t) \leq V_2(t)$ for $t \in [t_0, r_1)$, the process is similar and thus omitted.

From the above discussion , we have that
\begin{align*}
(i) & \quad [a_1(|x^{(1)}(t)|) + a_2(|x^{(2)}(t)|)]/2 \leq V(t) \leq b_1(|x^{(1)}(t)|) + b_2(|x^{(2)}(t)|) \\
(ii) & \quad D^+V(t) \leq 0 \text{ if } V_1(t) \leq V_2(t) \text{ or } V_2(t) \leq V_1(t) ; t \geq t_0 \\
(iii) & \quad V(t_k) \leq (1 + d_k) V(t_k^-) , K \in Z^+
\end{align*}

We are now in a position to show that the uniform stability of the zero solution .
For any given $\delta > 0$ , let $M = \prod_{k=1}^{\infty} (1 + d_k)$ Then by
$\sum_{k=1}^{\infty} d_k < \infty$. We have $1 \leq M < \infty$. We choose $\delta$ such that
$\max\{b_1(\delta), b_2(\delta)\} \leq \min\{[a_1(\epsilon)]/4M, [a_2(\epsilon)]/4M\}$

Let $x(t) = x(t, t_0, x_0)$ denote the solution of (1) with $\|x_0\| \leq \delta$
Let $t_0 \in [t_{m-1}, t_m]$ , $m \in Z^+$ , it is obvious that
$V(t) \leq b_1(\delta) + b_2(\delta) \quad , \quad t \geq t_0$

Furthermore , we show
$V(t) \leq (1 + d_m)[b_1(\delta) + b_2(\delta)] , \quad t_m \leq t \leq t_{m+1}$
Otherwise , there must be a $r_3 \in (t_m, t_{m+1})$ such that
$V(r_3) = (1 + d_m)[b_1(\delta) + b_2(\delta)] , V(r_3) \geq V(r_3+t) , \text{ and } D^+V(r_3) > 0.$
By condition (ii) , we get $D^+V(r_3) \leq 0$, which is a contradiction and hence is valid.
From condition (ii) we arrive at
\[ V(t_{m+1}) \leq (1 + d_{m+1})V(t_{m+1}) \leq (1 + d_m)(1 + d_{m+1})[b_1(\delta) + b_2(\delta)] \]

In a similar way we can prove that
\[ V(t) \leq \prod_{i=m}^{m+k} (1 + d_i) [b_1(\delta) + b_2(\delta)] \quad t_0 \leq t \leq t_{k+k} \]

Let \( k \to \infty \) in the above inequality so that we get
\[ V(t) \leq M [b_1(\delta) + b_2(\delta)] < a(\varepsilon), \quad t \geq t_0 \]

Which together with (ii) yields
\[ \left[ a_1\left(\frac{|x^{(1)}(t)|}{x^{(1)}(t)}\right) + a_2\left(\frac{|x^{(2)}(t)|}{x^{(2)}(t)}\right)\right]/2 \leq V(t) \leq M [b_1(\delta) + b_2(\delta)] \leq \min \{[a_1(\varepsilon)]/2, [a_2(\varepsilon)]/2\} \]

and thus
\[ a_1\left(\frac{|x^{(1)}(t)|}{x^{(1)}(t)}\right) \leq a_1(\varepsilon) \quad t \geq t_0 \]
\[ a_2\left(\frac{|x^{(2)}(t)|}{x^{(2)}(t)}\right) \leq a_2(\varepsilon) \quad t \geq t_0 \]

It follows that
\[ |x^{(1)}(t)| \leq \varepsilon \quad t \geq t_0 \]
\[ |x^{(2)}(t)| \leq \varepsilon \quad t \geq t_0 \]

Thus we have \[ |x(t)| \leq \varepsilon \quad t \geq t_0 \]

This completes the proof.

**References**


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