Some Growth Properties of Differential Polynomials

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Abstract

The aim of this paper is to study the comparative growth properties of composite entire or meromorphic functions and differential polynomials generated by one of the factors.

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1 Introduction, Definitions and Notations.

For any two transcendental entire functions $f$ and $g$ defined in the open complex plane $\mathbb{C}$, Clunie [5] proved that $\lim_{r \to \infty} \frac{T(r, f \circ g)}{T(r, f)} = \lim_{r \to \infty} \frac{T(r, f \circ g)}{T(r, g)} = \infty$. Singh [13] proved some comparative growth properties of $\log T(r, f \circ g)$ and $T(r, f)$. He [13] also raised the question of investigating the comparative growth of $\log T(r, f \circ g)$ and $T(r, g)$, which he was unable to solve. Lahiri[9] proved some results on the comparative growth of $\log T(r, g)$ and $T(r, f)$.

Let $f$ be a non-constant meromorphic function defined in the open complex plane $\mathbb{C}$. Also let $n_{0j}, n_{1j}, ..., n_{kj}$ ($k \geq 1$) be non-negative integers such that for each $j$, $\sum_{i=0}^{k} n_{ij} \geq 1$. We call $M_j[f] = A_j(f)^{n_{0j}}(f^{(1)})^{n_{1j}}....(f^{(k)})^{n_{kj}}$ where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by $f$. The
numbers \( \gamma_{M_j} = \sum_{i=0}^{k} n_{ij} \) and \( \Gamma_{M_j} = \sum_{i=0}^{k} (i + 1) n_{ij} \) are called respectively the degree and the weight of \( M_j[f] \). The expression \( P[f] = \sum_{j=1}^{s} M_j[f] \) is called a differential polynomial generated by \( f \). The numbers \( \gamma_P = \max_{1<j<s} \gamma_{M_j} \) and \( \Gamma_P = \max_{1<j<s} \Gamma_{M_j} \) are called respectively the degree and weight of \( P[f] \). Also we call the numbers \( \gamma_P - \gamma_{P_0} = \min_{1<j<s} \gamma_{M_j} \) and \( k \) (the order of the highest derivative of \( f \)) the lower degree and the order of \( P[f] \) respectively. If \( \gamma_P = \gamma_P, P[f] \) is called a homogeneous differential polynomial. In the paper we establish some newly developed results based on the comparative growth properties of composite entire or meromorphic functions and differential polynomials generated by one of the factors. Throughout the paper we consider only the non-constant differential polynomials and we denote by \( P_0[f] \) a differential polynomial not containing \( f \) i.e. for which \( n_{ij} = 0 \) for \( j = 1, 2, \ldots, s \). We consider only those \( P[f], P_0[f] \) singularities of whose individual terms do not cancel each other. We do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [14] and [8].

The following definitions are well known.

**Definition 1.** The order \( \rho_f \) and lower order \( \lambda_f \) of a meromorphic function \( f \) are defined as

\[
\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.
\]

If \( f \) is entire, one can easily verify that

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^2 M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^2 M(r, f)}{\log r},
\]

where \( \log^k x = \log(\log^{k-1} x) \) for \( k = 1, 2, 3, \ldots \) and \( \log^0 x = x \).

**Definition 2.** The hyper order \( \tilde{\rho}_f \) and hyper lower order \( \tilde{\lambda}_f \) of a meromorphic function \( f \) are defined as follows

\[
\tilde{\rho}_f = \limsup_{r \to \infty} \frac{\log^3 T(r, f)}{\log r} \quad \text{and} \quad \tilde{\lambda}_f = \liminf_{r \to \infty} \frac{\log^3 T(r, f)}{\log r}.
\]

If \( f \) is entire, then

\[
\tilde{\rho}_f = \limsup_{r \to \infty} \frac{\log^3 M(r, f)}{\log r} \quad \text{and} \quad \tilde{\lambda}_f = \liminf_{r \to \infty} \frac{\log^3 M(r, f)}{\log r}.
\]
Definition 3. [11] Let $f$ be a meromorphic function of order zero. Then the quantities $\rho^*_f, \lambda^*_f$ and $\bar{\rho}^*_f, \bar{\lambda}^*_f$ are defined in the following way:

$$\rho^*_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log^2 r}, \lambda^*_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log^2 r}$$

and

$$\bar{\rho}^*_f = \limsup_{r \to \infty} \frac{\log[T(r, f)]}{\log^2 r}, \bar{\lambda}^*_f = \liminf_{r \to \infty} \frac{\log[T(r, f)]}{\log^2 r}.$$

If $f$ is entire then clearly

$$\rho^*_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{\log^2 r}, \lambda^*_f = \liminf_{r \to \infty} \frac{\log M(r, f)}{\log^2 r}$$

and

$$\bar{\rho}^*_f = \limsup_{r \to \infty} \frac{\log^3 M(r, f)}{\log^2 r}, \bar{\lambda}^*_f = \liminf_{r \to \infty} \frac{\log^3 M(r, f)}{\log^2 r}.$$

Definition 4. The type $\sigma_f$ of a meromorphic function $f$ is defined as

$$\sigma_f = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$ 

When $f$ is entire, then

$$\sigma_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$ 

Definition 5. The quantity $\Theta(a; f)$ of a meromorphic function $f$ is defined as follows

$$\Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{\bar{N}(r, a; f)}{T(r, f)}.$$ 

Definition 6. [10] For $a \in \mathbb{C} \cup \{\infty\}$, let $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \leq r$, where a zero of multiplicity $\leq p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly $p$ times; and $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way. We define

$$\delta_p(a; f) = 1 - \limsup_{r \to \infty} \frac{N_p(r, a; f)}{T(r, f)}.$$ 

Definition 7. [4] $P[f]$ is said to be admissible if

(i) $P[f]$ is homogeneous, or

(ii) $P[f]$ is non homogeneous and $m(r, f) = S(r, f)$. 

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [5] If \( f \) and \( g \) be two entire functions then for all sufficiently large values of \( r \),

\[
M(r, f \circ g) \geq M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right).
\]

Lemma 2. [1] If \( f \) is meromorphic and \( g \) is entire then for all sufficiently large values of \( r \),

\[
T(r, f \circ g) \leq \left\{1 + o(1)\right\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).
\]

Lemma 3. [3] Let \( f \) be meromorphic and \( g \) be entire and also suppose that \( 0 < \mu \leq \rho_g \leq \infty \). Then for a sequence of values of \( r \) tending to infinity,

\[
T(r, f \circ g) \geq T(\exp(r^\mu), f).
\]

Lemma 4. [4] Let \( P_0[f] \) be admissible. If \( f \) is of finite order or of non zero lower order and \( \sum_{a \neq \infty} \Theta(a; f) = 2 \) then

\[
\lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \Gamma_{P_0}.
\]

Lemma 5. [4] Let \( f \) be either of finite order or of non zero lower order such that \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \). Then for homogeneous \( P_0[f] \),

\[
\lim_{r \to \infty} \frac{T(r, P_0[f])}{T(r, f)} = \gamma_{P_0}.
\]

Lemma 6. Let \( f \) be a meromorphic function of finite order or of non zero lower order. If \( \sum_{a \neq \infty} \Theta(a; f) = 2 \), then the order (lower order) of homogeneous \( P_0[f] \) is same as that of \( f \). Also the type of \( P_0[f] \) is \( \Gamma_{P_0} \) times that of \( f \).

**Proof.** By Lemma 4, \( \lim_{r \to \infty} \frac{\log T(r, P_0[f])}{\log T(r, f)} \) exists and is equal to 1. So

\[
\rho_{P_0[f]} = \lim_{r \to \infty} \sup \frac{\log T(r, P_0[f])}{\log r}
\]

\[
= \lim_{r \to \infty} \sup \frac{\log T(r, f) \lim_{r \to \infty} \frac{\log T(r, P_0[f])}{\log T(r, f)}}{\log r}
\]

\[
= \rho_f \cdot 1 = \rho_f.
\]
In a similar manner, \( \lambda_{P_0[f]} = \lambda_f \).
Again by Lemma 4,

\[
\sigma_{P_0[f]} = \lim_{r \to \infty} \frac{T(r, P_0[f])}{r^{\rho_{P_0[f]}}} = \lim_{r \to \infty} \frac{T(r, f)}{T(r, f)} = \Gamma_{P_0} \sigma_f.
\]

This proves the lemma.

**Lemma 7.** Let \( f \) be a meromorphic function of finite order or of non zero lower order such that \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \). Then the order (lower order) of homogeneous \( P_0[f] \) and \( f \) are same and the type of \( P_0[f] \) is \( \gamma_{P_0} \) times that of \( f \).
We omit the proof of the lemma because it can be carried out in the line of Lemma 6 and with the help of Lemma 5.
In a similar manner we can state the following lemma without proof.

**Lemma 8.** Let \( f \) be a meromorphic function of finite order or of non zero lower order such that \( \delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1 \). Then for every homogeneous \( P_0[f] \) the order (lower order) of \( P_0[f] \) is same as that of \( f \). Also the type of \( P_0[f] \) is \( \gamma_{P_0} \) times that of \( f \).

**Lemma 9.** Let \( f \) be a meromorphic function of finite order or of non zero lower order and \( \sum_{a \neq \infty} \Theta(a; f) = 2 \). Then the hyper order (hyper lower order) of \( P_0[f] \) and \( f \) are equal.
We omit the proof of Lemma 9 because it can be carried out in the line of Lemma 6.

**Lemma 10.** [7] Let \( f \) be meromorphic and \( g \) be transcendental entire. If \( \rho_{f \circ g} < \infty \) then \( \rho_f = 0 \).

**Remark 1.** The conclusion of Lemma 9 can also be deduced under the hypothesis \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \) or \( \delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1 \) instead of \( \sum_{a \neq \infty} \Theta(a; f) = 2 \).

**Lemma 11.** Let \( f \) be meromorphic and \( g \) be transcendental entire such that \( \rho_f = 0 \) and \( \rho_g < \infty \). Then \( \rho_{f \circ g} \leq \rho^*_f \rho_g \).

**Proof.** In view of Lemma 2 and the inequality \( T(r, g) \leq \log^+ M(r, g) \), we get

\[
\rho_{f \circ g} = \lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log r}
\]
\[
\frac{\log T(M(r, g), f) + o(1)}{\log r} \leq \limsup_{r \to \infty} \log T(M(r, g), f) + o(1)
\]
\[
\leq \limsup_{r \to \infty} \frac{\log T(M(r, g), f)}{\log r} \limsup_{r \to \infty} \frac{\log^2 M(r, g)}{\log r}
\]
\[
= \rho_f^* \rho_g.
\]
This proves the lemma.

### 3 Theorems.

In this section we present the main results of the paper.

**Theorem 1.** Let \( f \) be transcendental meromorphic and \( g \) be entire satisfying the following conditions:

(i) \( \rho_f \) and \( \rho_g \) are both finite

(ii) \( \rho_f \) is positive and

(iii) \( \sum_{a \neq \infty} \Theta(a; f) = 2 \).

Then for each \( \alpha \in (-\infty, \infty) \),

\[
\liminf_{r \to \infty} \frac{\{\log T(r, f \circ g)\}^{1+\alpha}}{\log T(\exp(r^{p'}), P_0[f])} = 0 \quad \text{if} \quad p' > (1+\alpha)\rho_g.
\]

**Proof.** If \( 1 + \alpha \leq 0 \), the theorem is trivial. So we take \( 1 + \alpha > 0 \).

Since \( T(r, g) \leq \log^+ M(r, g) \), by Lemma 2 we get for all sufficiently large values of \( r \),

\[
T(r, f \circ g) \leq \{1 + o(1)\} T(M(r, g), f)
\]

i.e., \( \log T(r, f \circ g) \leq \log\{1 + o(1)\} + \log T(M(r, g), f) \)

i.e., \( \log T(r, f \circ g) \leq \{1 + o(1)\} + (\rho_f + \epsilon) \log M(r, g) \)

i.e., \( \log T(r, f \circ g) \leq \{1 + o(1)\} + (\rho_f + \epsilon) r^{p_g + \epsilon} \)

i.e., \( \log T(r, f \circ g) \leq r^{p_g + \epsilon} \{ (\rho_f + \epsilon) + o(1) \} \)

i.e., \( \{\log T(r, f \circ g)\}^{1+\alpha} \leq r^{(p_g + \epsilon)(1+\alpha)} \{ (\rho_f + \epsilon) + o(1) \}^{1+\alpha} \).

(1)
Again we have for a sequence of values of \( r \) tending to infinity and for \( \epsilon > 0 \),

\[
\log T(\exp(r^p), P_0[f]) > (\rho_{P_0[f]} - \epsilon) \log(\exp(r^p)). \tag{2}
\]

Now combining (1) and (2) and in view of Lemma 6 we obtain for a sequence of values of \( r \) tending to infinity,

\[
\frac{\log T(r, f \circ g)\{1+\alpha\}}{\log T(\exp(r^p), P_0[f])} \leq \frac{r^{(\rho_g+\epsilon)\{1+\alpha\}}\{(\rho_f + \epsilon) + o(1)\}^{1+\alpha}}{(\rho_f - \epsilon)r^p},
\]

from which the theorem follows because we can choose \( \epsilon \) such that \( 0 < \epsilon < \min\{\rho_f, \frac{r}{1+\alpha} - \rho_g\} \).

This proves the theorem.

**Remark 2.** The conclusion of Theorem 1 can also be drawn under the hypothesis \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \) or \( \delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1 \) instead of \( \sum_{a \neq \infty} \Theta(a; f) = 2 \).

**Theorem 2.** If \( f \) be meromorphic and \( g \) be transcendental entire such that \( \rho_g < \infty \), \( \rho_{f \circ g} = \infty \) and \( \Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1 \), then for every \( A > 0 \),

\[
\limsup_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, P_0[g])} = \infty.
\]

**Proof.** If possible, let there exists a constant \( \beta \) such that for all sufficiently large values of \( r \), we have

\[
\log T(r, f \circ g) \leq \beta \log T(r^A, P_0[g]). \tag{3}
\]

In view of Lemma 7, for all sufficiently large values of \( r \) we get

\[
\log T(r^A, P_0[g]) \leq (\rho_{P_0[g]} + \epsilon)A \log r
\]

i.e., \( \log T(r^A, P_0[g]) \leq (\rho_g + \epsilon)A \log r. \tag{4} \)

Now combining (3) and (4) we obtain for all sufficiently large values of \( r \),

\[
\log T(r, f \circ g) \leq \beta(\rho_g + \epsilon)A \log r
\]

i.e., \( \rho_{f \circ g} \leq \beta A(\rho_g + \epsilon) \),

which contradicts the condition \( \rho_{f \circ g} = \infty \).

So for a sequence of values of \( r \) tending to infinity, it follows that

\[
\log T(r, f \circ g) > \beta \log T(r^A, P_0[g]),
\]

from which the theorem follows.
Corollary 1. Under the assumptions of Theorem 2,
\[ \limsup_{r \to \infty} \frac{T(r, f \circ g)}{T(r^A, P_0[g])} = \infty. \]

Proof. By Theorem 2 we obtain for all sufficiently large values of \( r \) and for \( K > 1 \),
\[ \log T(r, f \circ g) > K \log T(r^A, P_0[g]) \]
i.e., \( T(r, f \circ g) > \{T(r^A, P_0[g]) \}^K \), from which the corollary follows.

Remark 3. If we take \( \rho_f < \infty \) and \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) \) instead of \( \rho_g < \infty \) and \( \Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) \) respectively, then Theorem 2 and Corollary 1 remain valid with \( P_0[g] \) replaced by \( P_0[f] \) in the denominator.

Theorem 3. Let \( f \) and \( g \) be two entire functions with \( \lambda_f > 0 \) and \( \rho_f < \lambda_g \). Also let \( f \) be transcendental with \( \sum_{a \neq \infty} \Theta(a; f) = 2 \). Then
\[ \lim_{r \to \infty} \frac{\log M(r, f \circ g)}{\log M(r, P_0[f])} = \infty. \]

Proof. In view of Lemma 1, we have for all sufficiently large values of \( r \),
\[ M(r, f \circ g) \geq M\left( \frac{1}{16} M\left( \frac{r}{2}, g \right), f \right) \]
i.e., \( \log M(r, f \circ g) \geq \log M\left( \frac{1}{16} M\left( \frac{r}{2}, g \right), f \right) \)
i.e., \( \log M(r, f \circ g) \geq (\lambda_f - \epsilon) \log \left( \frac{1}{16} M\left( \frac{r}{2}, g \right) \right) \)
i.e., \( \log M(r, f \circ g) \geq O(1) + (\lambda_f - \epsilon) \left( \frac{r}{2} \right)^{\lambda_f - \epsilon}. \) \hspace{1cm} (5)
Again for all sufficiently large values of \( r \), we get by Lemma 6
\[ \log M(r, P_0[f]) \leq r^{(\rho_{P_0[f]} + \epsilon)} = r^{(\rho_f + \epsilon)}. \] \hspace{1cm} (6)
Now combining (5) and (6) it follows for all sufficiently large values of \( r \),
\[
\frac{\log^2 \mu(r, f \circ g)}{\log \mu(r, P_0[f])} \geq \frac{O(1) + (\lambda_f - \epsilon)(\frac{r}{2})^{(\lambda_g + \epsilon)}}{r^{(\rho_f + \epsilon)}}.
\]
(7)
Since \( \rho_f < \lambda_g \), we can choose \( \epsilon(>0) \) in such a way that
\[
\rho_f + \epsilon < \lambda_g - \epsilon.
\]
(8)
Thus from (7) and (8) we obtain that
\[
\lim_{r \to \infty} \frac{\log^2 \mu(r, f \circ g)}{\log \mu(r, P_0[f])} = \infty,
\]
from which the theorem follows.

**Theorem 4.** If \( f \) be a transcendental meromorphic function and \( g \) be entire
with \( 0 < \lambda_f \leq \rho_f < \infty, \rho_g < \infty \) and \( \delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1 \), then
\[
\lim_{r \to \infty} \frac{T(r, f \circ g)T(r, P_0[f])}{T(\exp(r^p), P_0[f])} = 0, \text{ if } p' > \rho_g.
\]
**Proof.** Since \( T(r, g) \leq \log^+ M(r, g) \), for all sufficiently large values of \( r \) we get from Lemma 2,
\[
T(r, f \circ g) \leq \{1 + o(1)\}T(M(r, g), f)
\]
i.e.,
\[
T(r, f \circ g) \leq \{1 + o(1)\}\exp\{(\rho_f + \epsilon)r^{(\rho_g + \epsilon)}\}.
\]
(9)
Again by Lemma 8 we obtain for all sufficiently large values of \( r \),
\[
T(r, P_0[f]) \leq r^{(\rho_{P_0[f]} + \epsilon)} = r^{\rho_f + \epsilon}.
\]
(10)
Now combining (9) and (10) it follows for all sufficiently large values of \( r \),
\[
T(r, f \circ g)T(r, P_0[f]) \leq \{1 + o(1)\}r^{\rho_f + \epsilon}\exp\{(\rho_f + \epsilon)r^{(\rho_g + \epsilon)}\}.
\]
(11)
Also in view of Lemma 8 we have for all sufficiently large values of \( r \),
\[
\log T(\exp(r^p), P_0[f]) \geq (\lambda_{P_0[f]} - \epsilon)\log(\exp(r^p))
\]
i.e.,
\[
T(\exp(r^p), P_0[f]) \geq \exp\{(\lambda_f - \epsilon)r^p\}.
\]
(12)
From (11) and (12) it follows for all sufficiently large values of \( r \),
\[
\frac{T(r, f \circ g)T(r, P_0[f])}{T(\exp(r^p), P_0[f])} \leq \frac{\{1 + o(1)\}r^{\rho_f + \epsilon}\exp\{(\rho_f + \epsilon)r^{(\rho_g + \epsilon)}\}}{\exp\{(\lambda_f - \epsilon)r^p\}}.
\]
(13)
As \( p' > \rho_g \) so we can choose \( \epsilon(>0) \) such that
\[
p' > \rho_g + \epsilon.
\]
(14)
Thus the theorem follows from (13) and (14).
Theorem 5. Let \( f \) be a transcendental meromorphic function and \( g \) be a transcendental entire function such that \( 0 < \lambda_f \leq \rho_f < \infty \) and \( \sum_{a \neq \infty} \Theta(a; f) = 2 \). Then for every \( A > 0 \),

\[
\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, P_0[f])} = \infty.
\]

If further, \( \rho_g < \infty \) and \( \sum_{a \neq \infty} \Theta(a; g) = 2 \) then

\[
\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, P_0[g])} = \infty.
\]

Proof. Since \( \lambda_f > 0, \lambda_{fog} = \infty \) (cf. [2]). So it follows that for arbitrary large \( N \) and for all large values of \( r \)

\[
\log T(r, f \circ g) > A_N \log r. \tag{15}
\]

Again since \( \rho_f < \infty \), for all large values of \( r \) we get by Lemma 6,

\[
\log T(r^A, P_0[f]) \leq A(\rho_f + 1) \log r. \tag{16}
\]

Now from (15) and (16) it follows for all large values of \( r \) that,

\[
\frac{\log T(r, f \circ g)}{\log T(r^A, P_0[f])} > \frac{A_N \log r}{A(\rho_f + 1) \log r}
\]

and so

\[
\lim_{r \to \infty} \frac{\log T(r, f \circ g)}{\log T(r^A, P_0[f])} = \infty.
\]

Again since \( \rho_g < \infty \) then for all large values of \( r \) we obtain by Lemma 6

\[
\log T(r^A, P_0[g]) < A(\rho_g + 1) \log r. \tag{17}
\]

Now from (15) and (17) it follows for all large values of \( r \) that

\[
\frac{\log T(r, f \circ g)}{\log T(r^A, P_0[g])} > \frac{A_N \log r}{A(\rho_g + 1) \log r}. \tag{18}
\]

Thus the theorem follows from (18).

Theorem 6. Let \( f \) be a meromorphic function with \( \lambda_f > 0 \) and \( g \) be transcendental entire satisfying \( \rho_f < \infty \) and \( \Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1 \).

Then

\[
\limsup_{r \to \infty} \frac{\log T(\exp(r^\mu), f \circ g)}{\log T(\exp(r^\mu), P_0[f])} = \infty \quad \text{where} \quad 0 < \mu < \rho_g.
\]
Proof. Let $0 < \mu' < \rho_g$. Then in view of Lemma 3, we get for a sequence of values of $r$ tending to infinity,

$$\log T(r, f \circ g) \geq \log T(\exp(r\mu'), f)$$

i.e., \(\log T(r, f \circ g) \geq (\lambda_f - \epsilon) \log \{\exp(r\mu')\}\)

i.e., \(\log T(r, f \circ g) \geq (\lambda_f - \epsilon)r^{\mu'}\)

i.e., \(\log^{[2]} T(r, f \circ g) \geq O(1) + \mu' \log r\).

So for a sequence of values of $r$ tending to infinity,

$$\log^{[2]} T(\exp(r^{\rho_g}), f \circ g) \geq O(1) + \mu' \log \{\exp(r^{\rho_g})\}\)

i.e., \(\log^{[2]} T(\exp(r^{\rho_g}), f \circ g) \geq O(1) + \mu' r^{\rho_g}. \quad (19)\)

Again in view of Lemma 7, we obtain for all sufficiently large values of $r$,

$$\log T(\exp(r^{\mu'}), P_0[f]) \leq (\rho_{P_0[f]} + \epsilon) \log \{\exp(r^{\mu'})\}\)

i.e., \(\log T(\exp(r^{\mu'}), P_0[f]) \leq (\rho_f + \epsilon)r^{\mu'}\). \quad (20)\)

Combining (19) and (20) it follows for a sequence of values of $r$ tending to infinity,

$$\frac{\log^{[2]} T(\exp(r^{\rho_g}), f \circ g)}{\log T(\exp(r^{\mu'}), P_0[f])} \geq \frac{O(1) + \mu' r^{\rho_g}}{(\rho_f + \epsilon)r^{\mu'}}. \quad (21)\)

Since $\mu < \rho_g$, we get from (21) that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f \circ g)}{\log T(\exp(r^{\mu'}), P_0[f])} = \infty.$$

This proves the theorem.

Theorem 7. Let $f$ be rational and $g$ be transcendental meromorphic satisfying \(0 < \overline{\lambda}_{fg} \leq \overline{\rho}_{fg} < \infty, 0 < \lambda_g \leq \overline{\rho}_g < \infty \) and \(\sum_{a \neq \infty} \Theta(a; g) = 2\). Then for any positive number $A$,

$$\frac{\overline{\lambda}_{fg}}{A\overline{\rho}_g} \leq \liminf_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^{A}, P_0[g])} \leq \frac{\overline{\lambda}_{fg}}{A\lambda_g}$$

$$\leq \limsup_{r \to \infty} \frac{\log^{[2]} T(r, f \circ g)}{\log^{[2]} T(r^{A}, P_0[g])} \leq \frac{\overline{\rho}_{fg}}{A\lambda_g}.$$
**Proof.** From the definition of hyper order and hyper lower order and by Lemma 9, we get for arbitrary positive $\epsilon$ and for all sufficiently large values of $r$,

$$\log^2 T(r, f \circ g) \geq (\overline{\lambda}_{f \circ g} - \epsilon) \log r \quad (22)$$

and

$$\log^2 T(r^A, P_0[g]) \leq (\overline{\rho}_{P_0[g]} + \epsilon) \log r^A \quad (23)$$

i.e., $\log^2 T(r^A, P_0[g]) \leq A(\overline{\rho}_g + \epsilon) \log r$.

Combining (22) and (23), we obtain for all sufficiently large values of $r$,

$$\frac{\log^2 T(r, f \circ g)}{\log^2 T(r^A, P_0[g])} \geq \frac{(\overline{\lambda}_{f \circ g} - \epsilon) \log r}{A(\overline{\rho}_g + \epsilon) \log r}.$$ 

Since $\epsilon(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r \to \infty} \frac{\log^2 T(r, f \circ g)}{\log^2 T(r^A, P_0[g])} \geq \frac{\overline{\lambda}_{f \circ g}}{A\overline{\rho}_g}. \quad (24)$$

Again for a sequence of values of $r$ tending to infinity,

$$\log^2 T(r, f \circ g) \geq (\overline{\lambda}_{f \circ g} + \epsilon) \log r. \quad (25)$$

Also in view of Lemma 9, we have for all sufficiently large values of $r$,

$$\log^2 T(r^A, P_0[g]) \leq (\overline{\lambda}_{P_0[g]} - \epsilon) \log r^A \quad (26)$$

i.e., $\log^2 T(r^A, P_0[g]) \leq A(\overline{\lambda}_g - \epsilon) \log r$.

Combining (25) and (26) we get for a sequence of values of $r$ tending to infinity,

$$\frac{\log^2 T(r, f \circ g)}{\log^2 T(r^A, P_0[g])} \leq \frac{(\overline{\lambda}_{f \circ g} + \epsilon)}{A(\overline{\lambda}_g - \epsilon)}.$$ 

As $\epsilon(> 0)$ is arbitrary it follows from above that

$$\liminf_{r \to \infty} \frac{\log^2 T(r, f \circ g)}{\log^2 T(r^A, P_0[g])} \leq \frac{\overline{\lambda}_{f \circ g}}{A\overline{\lambda}_g}. \quad (27)$$

Also for a sequence of values of $r$ tending to infinity and by Lemma 9,

$$\log^2 T(r^A, P_0[g]) \leq A(\overline{\lambda}_{P_0[g]} + \epsilon).$$
i.e., $\log^2 T(r^A, P_0[g]) \leq A(\overline{\lambda}_g + \epsilon)$. \hfill (28)

Combining (22) and (28) we have for a sequence of values of $r$ tending to infinity,

$$\frac{\log^2 T(r, f \circ g)}{\log^2 T(r^A, P_0[g])} \geq \frac{(\overline{\lambda}_{fog} - \epsilon) \log r}{A(\overline{\lambda}_g + \epsilon) \log r}. \hfill (29)$$

Since $\epsilon(>0)$ is arbitrary it follows from above that

$$\limsup_{r \to \infty} \frac{\log^2 T(r, f \circ g)}{\log^2 T(r^A, P_0[g])} \geq \frac{\overline{\lambda}_{fog}}{A\overline{\lambda}_g}. \hfill (30)$$

Also for all sufficiently large values of $r$,

$$\log^2 T(r, f \circ g) \leq (\overline{\rho}_{fog} + \epsilon) \log r. \hfill (31)$$

From (26) and (30) we obtain for all sufficiently large values of $r$,

$$\frac{\log^2 T(r, f \circ g)}{\log^2 T(r^A, P_0[g])} \leq \frac{(\overline{\rho}_{fog} + \epsilon) \log r}{A(\overline{\lambda}_g - \epsilon) \log r}. \hfill (32)$$

As $\epsilon(>0)$ is arbitrary it follows from above that

$$\limsup_{r \to \infty} \frac{\log^2 T(r, f \circ g)}{\log^2 T(r^A, P_0[g])} \leq \frac{\overline{\rho}_{fog}}{A\overline{\lambda}_g}. \hfill (33)$$

Thus the theorem follows from (24), (27), (29) and (31).

**Theorem 8.** Let $f$ be meromorphic and $g$ be transcendental entire such that (i) $0 < \rho_g < \infty$, (ii) $\sigma_g > 0$, (iii) $0 < \rho_{fog} < \infty$, (iv) $\sigma_{fog} < \infty$, (v) $\rho_f^* < 1$ and (vi) $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$. Then

$$\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} = 0.$$

**Proof.** From the definition of type, we have for arbitrary positive $\epsilon$ and for sufficiently large values of $r$,

$$T(r, f \circ g) \leq (\sigma_{fog} + \epsilon)r^{\rho_{fog}}. \hfill (34)$$

Again in view of Lemma 9 we get for a sequence of values of $r$ tending to infinity,

$$T(r, P_0[g]) \geq (\sigma_{P_0[g]} - \epsilon)r^{\rho_{P_0[g]}}$$
i.e., \( T(r, P_0[g]) \geq (\gamma P_0, \sigma_g - \epsilon)r^{\rho_g}. \) \hspace{1cm} (33)

Since \( \rho_{fog} < \infty \), it follows that \( \rho_f = 0 \) [cf.[7]].
So in view of Lemma 11, from (32)and (33) we obtain for a sequence of values of \( r \) tending to infinity,

\[
\frac{T(r, f \circ g)}{T(r, P_0[g])} \leq \frac{(\sigma_{fog} + \epsilon)r^{\rho_{fog}}}{(\gamma P_0, \sigma_g - \epsilon)r^{\rho_g}}
\]

i.e., \( T(r, f \circ g) \leq \frac{(\sigma_{fog} + \epsilon)}{(\gamma P_0, \sigma_g - \epsilon)}r^{(\rho_{fog} - 1)\rho_g}. \)

Since \( \epsilon(> 0) \) is arbitrary, in view of condition \((v)\), it follows that
\[
\liminf_{r \to \infty} \frac{T(r, f \circ g)}{T(r, P_0[g])} = 0.
\]

This proves the theorem.

References


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