On a Quasilinear Pseudohyperbolic Equations with a Nonlocal Boundary Condition

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Abstract. The present paper is devoted to study initial-boundary value problems associating an integral condition with Neumann condition for the quasilinear pseudohyperbolic equations. We establish the existence, uniqueness and continuous dependence upon the data of weak solution for the quasilinear problem.

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1. Statement of the problem

In the present work, we are interested with a mixed problem with Neumann and integral conditions for a quasilinear viscosity equation

\( \frac{\partial^2 \theta}{\partial t^2} - \alpha \frac{\partial^2 \theta}{\partial x^2} - \beta \frac{\partial^3 \theta}{\partial t \partial x^2} = g \left( x, t, \theta, \frac{\partial \theta}{\partial t} \right), \quad (x, t) \in \Omega \times (0, T) \)  

(1.1)

\( \theta(x, 0) = \Phi(x), \quad \frac{\partial \theta(x, 0)}{\partial t} = \Psi(x), \quad x \in \Omega, \)  

(1.2)

\( \frac{\partial \theta(0, t)}{\partial x} = \mu(t), \quad \int_{\Omega} \theta(x, t) dx = m(t), \quad t \in I, \)  

(1.3)
We introduce a new function $u(x, t)$ representing the deviation of the function $\theta(x, t)$ from the known function $U(x, t)$ constructed in [4]. This function $u(x, t)$ will be defined as the solution of problem

\[
\frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial^3 u}{\partial t \partial x^2} = f \left( x, t, u, \frac{\partial u}{\partial t} \right), \quad (x, t) \in Q, 
\]

\[
u(x, 0) = \varphi(x), \quad \frac{\partial u(x, 0)}{\partial t} = \psi(x), \quad x \in \Omega, 
\]

\[
\frac{\partial u(0, t)}{\partial x} = 0, \quad \int_\Omega u(x, t)dx = 0, \quad t \in I.
\]

where

\[
f \left( x, t, u, \frac{\partial u}{\partial t} \right) = g \left( x, t, u + U, \frac{\partial u}{\partial t} + \frac{\partial U}{\partial t} \right) - \frac{\partial^2 U}{\partial t^2}.
\]

Let us consider the following auxiliary problem:

\[
\frac{\partial^2 w}{\partial t^2} - \alpha \frac{\partial^2 w}{\partial x^2} - \beta \frac{\partial^3 w}{\partial t \partial x^2} = 0
\]

\[
w(x, 0) = \varphi(x), \quad \frac{\partial w(x, 0)}{\partial t} = \psi(x),
\]

\[
\frac{\partial w(0, t)}{\partial x} = 0, \quad \int_\Omega w(x, t)dx = 0.
\]

Set

\[
z(x, t) = u(x, t) - w(x, t),
\]

then $z(x, t)$ satisfies:

\[
\frac{\partial^2 z}{\partial t^2} - \alpha \frac{\partial^2 z}{\partial x^2} - \beta \frac{\partial^3 z}{\partial t \partial x^2} = f \left( x, t, z + w, \frac{\partial z}{\partial t} + \frac{\partial w}{\partial t} \right),
\]

\[
z(x, 0) = 0, \quad \frac{\partial z(x, 0)}{\partial t} = 0,
\]

\[
\frac{\partial z(0, t)}{\partial x} = 0, \quad \int_\Omega z(x, t)dx = 0.
\]

From [6], we conclude that problem (1.7)-(1.9) possesses a unique weak solution that depends upon the initial conditions, thus it remains to prove the weak solvability of problem (1.10)-(1.12).

The plan of the paper is the following:

2. Notations and preliminary results
3. A priori estimates
4. Existence, uniqueness and continuous dependence of the solution.
2. Notation and Preliminary Results

The following notation will be used throughout the paper:

Let $H$ be a Hilbert space with norm $\| \cdot \|_H$. For simplicity, throughout the paper, $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$ and $\| \cdot \|$ its associated norm. By $L^2_0(\Omega)$, we denote the space of square integrable functions with zero mean value on $\Omega$, namely

\[(2.1)\quad L^2_0(\Omega) = \left\{ u \in L^2(\Omega) : \int_{\Omega} u \, dx = 0 \right\},\]

which is clearly a Hilbert space for the scalar product $(\cdot, \cdot)$. We denote by $L^2(I; H), H^1(I; H)$ the standard function spaces; by $C^0_0(\Omega)$ the space of continuous functions with compact support in $\Omega$; by $B^1_{1,2}(\Omega)$ the space of square summable primitive functions, first introduced by the author [1, 2, 5], which can be considered as a completion of space $C^0_0(\Omega)$ for the scalar product

\[(u, w)_{B^1_{1,2}(\Omega)} = \int_{\Omega} \Im_x u \, \Im_x w \, dx,\]

and the corresponding norm

\[\| u \|_{B^1_{1,2}(\Omega)} = \sqrt{(u, u)_{B^1_{1,2}(\Omega)}} = \| \Im_x u \|,\]

for every fixed $x \in \Omega$, where $\Im_x u = \int_0^x u(\xi) \, d\xi$. It is easy to get

\[(2.2)\quad \| u \|_{B^1_{1,2}(\Omega)}^2 \leq \frac{l^2}{2} \| u \|^2,\]

for every $u \in L^2(\Omega)$, from which we deduce the continuity of the imbedding $L^2(\Omega) \to B^1_{1,2}(\Omega)$.

Now, we consider the scalar product in $L^2(I, B^1_{1,2}(\Omega))$ of equation (1.4) and the test function $v(x, t) \in V = \{ v/v \in H^1(I, L^2_0(\Omega)) : v(., T) = 0 \}$, yields

\[
\begin{align*}
\left( \frac{\partial^2 u}{\partial t^2}, v \right)_{L^2(I, B^1_{1,2}(\Omega))} &- \alpha \left( \frac{\partial^2 u}{\partial x^2}, v \right)_{L^2(I, B^1_{1,2}(\Omega))} - \beta \left( \frac{\partial^3 u}{\partial t \partial x^2}, v \right)_{L^2(I, B^1_{1,2}(\Omega))} \\
&= \left( f \left( .., u, \frac{\partial u}{\partial t} \right), v \right)_{L^2(I, B^1_{1,2}(\Omega))}.
\end{align*}
\]

It then follows that

\[
\begin{align*}
- \left( \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right)_{L^2(I, B^1_{1,2}(\Omega))} &+ \alpha \left( u, v \right)_{L^2(I, L^2_0(\Omega))} - \beta \left( u, \frac{\partial v}{\partial t} \right)_{L^2(I, L^2_0(\Omega))} \\
&= \left( f \left( .., u, \frac{\partial u}{\partial t} \right), v \right)_{L^2(I, B^1_{1,2}(\Omega))} + \beta \left( \varphi, v(., 0) \right) + \beta \left( \psi, v(., 0) \right)_{B^1_{1,2}(\Omega)}.
\end{align*}
\]
The above equality can be written as follows:

\[
A(u, v) = \left( f\left(u, \frac{\partial u}{\partial t}\right), v \right)_{L^2(I, B^1_2(\Omega))} + \beta(\varphi, v(., 0)) + (\psi, v(., 0))_{B^1_2(\Omega)},
\]

where \(A(u, v)\) is its left-hand side.

We look for a weak solution in the following sense:

**Definition 1.** The function \(u\) is called a weak solution of problem (1.4)-(1.6), if the following properties are fulfilled:

i) \(u \in L^2(I, L^2_0(\Omega))\),

ii) \(\frac{\partial u}{\partial t} \in L^2(I, B^1_2(\Omega))\),

iii) \(u\) verifies the integral condition (1.6),

iv) the integral identity (2.3) holds for all functions \(v \in V\).

We observe that due to i), the integral condition (1.6) has sense, and by virtue of i) and ii) and (1.7) each term in the integral identity (2.3) is well defined.

### 3. A priori estimates

We shall proceed by iteration as follows:

Start with \(z^{(0)}(x, t) \equiv 0, \forall (x, t) \in Q\), we define the sequence \(\{z^{(n)}\}_{n\in\mathbb{N}}\) by:

If \(z^{(n-1)}\) is known, then find \(z^{(n)}\) solution, in a weak sense, of problem:

\[
\frac{\partial^2 z^{(n)}}{\partial t^2} - \alpha \frac{\partial^2 z^{(n)}}{\partial x^2} - \beta \frac{\partial^3 z^{(n)}}{\partial t \partial x^2} = f\left(x, t, z^{(n-1)} + w, \frac{\partial z^{(n-1)}}{\partial t}, \frac{\partial w}{\partial t}\right),
\]

\[
z^{(n)}(x, 0) = 0, \quad \frac{\partial z^{(n)}}{\partial t}(x, 0) = 0,
\]

\[
\frac{\partial z^{(n)}}{\partial x}(0, t) = 0, \quad \int_{\Omega} z^{(n)}(x, t)dx = 0.
\]

The paper [6] implies, for fixed \(n\), that each of problems (3.1)-(3.3) admits a unique solution. Set

\[\zeta^{(n)} = z^{(n)} - z^{(n-1)},\]

then \(\zeta^{(n)}\) satisfies

\[
\frac{\partial^2 \zeta^{(n)}}{\partial t^2} - \alpha \frac{\partial^2 \zeta^{(n)}}{\partial x^2} - \beta \frac{\partial^3 \zeta^{(n)}}{\partial t \partial x^2} = f^{(n-1)}(x, t),
\]

\[
\zeta^{(n)}(x, 0) = 0, \quad \frac{\partial \zeta^{(n)}}{\partial t}(x, 0) = 0,
\]

\[
\frac{\partial \zeta^{(n)}}{\partial x}(0, t) = 0, \quad \int_{\Omega} \zeta^{(n)}(x, t)dx = 0.
\]
where
\[
\mathcal{F}^{(n-1)}(x, t) = f\left(x, t, z^{(n-1)} + w, \frac{\partial z^{(n-1)}}{\partial t} + \frac{\partial w}{\partial t}\right) - f\left(x, t, z^{(n-2)} + w, \frac{\partial z^{(n-2)}}{\partial t} + \frac{\partial w}{\partial t}\right).
\]

**Theorem 1.** Suppose that \( f(x, t, 0, 0) \in L^2(I, B_{2}^{1}(\Omega)) \) and \( f(x, t, p, q) \) verifies the Lipschitz condition; i.e.,
\[
\exists L > 0 : \|f(\cdot, p_1, q_1) - f(\cdot, p_2, q_2)\|_{B_{2}^{1}(\Omega)} \leq L \left(\|p_1 - p_2\|_{B_{2}^{1}(\Omega)} + \|q_1 - q_2\|_{B_{2}^{1}(\Omega)}\),
\]
then the following estimate holds:
\[
\|\zeta^{(n)}\|_{H_{1}^{1}(I, B_{2}^{1}(\Omega))} \leq c_4 \|\zeta^{(n-1)}\|_{H_{1}^{1}(I, B_{2}^{1}(\Omega))}, \quad n = 1, 2, ..., \]
where
\[
c_4 = \frac{L^2}{2\beta \min\{l^2, 2\alpha\}}.
\]

**Proof.** Taking the scalar product, in \( B_{2}^{1}(\Omega) \), of equation (3.4) and \( \frac{\partial \zeta^{(n)}}{\partial t} \), we have
\[
\left(\frac{\partial^2 \zeta^{(n)}(\cdot, t)}{\partial t^2}, \frac{\partial \zeta^{(n)}(\cdot, t)}{\partial t}\right)_{B_{2}^{1}(\Omega)} - \alpha \left(\frac{\partial^2 \zeta^{(n)}(\cdot, t)}{\partial x^2}, \frac{\partial \zeta^{(n)}(\cdot, t)}{\partial t}\right)_{B_{2}^{1}(\Omega)}
- \beta \left(\frac{\partial^2 \zeta^{(n)}(\cdot, t)}{\partial t \partial x^2}, \frac{\partial \zeta^{(n)}(\cdot, t)}{\partial t}\right)_{B_{2}^{1}(\Omega)}
= \left(\mathcal{F}^{(n-1)}(\cdot, t), \frac{\partial \zeta^{(n)}(\cdot, t)}{\partial t}\right)_{B_{2}^{1}(\Omega)}.
\]
Integrating by parts each term in the left-hand side of (3.9):
\[
\left(\frac{\partial^2 \zeta^{(n)}(\cdot, t)}{\partial t^2}, \frac{\partial \zeta^{(n)}(\cdot, t)}{\partial t}\right)_{B_{2}^{1}(\Omega)} = \frac{1}{2} \frac{\partial^2}{\partial t^2} \left\|\frac{\partial \zeta^{(n)}(\cdot, t)}{\partial t}\right\|^2_{B_{2}^{1}(\Omega)},
\]
\[
- \alpha \left(\frac{\partial^2 \zeta^{(n)}(\cdot, t)}{\partial x^2}, \frac{\partial \zeta^{(n)}(\cdot, t)}{\partial t}\right)_{B_{2}^{1}(\Omega)} = \frac{\alpha}{2} \frac{\partial}{\partial t} \left\|\zeta^{(n)}(\cdot, t)\right\|^2,
\]
\[
- \beta \left(\frac{\partial^2 \zeta^{(n)}(\cdot, t)}{\partial t \partial x^2}, \frac{\partial \zeta^{(n)}(\cdot, t)}{\partial t}\right)_{B_{2}^{1}(\Omega)} = \beta \left\|\frac{\partial \zeta^{(n)}(\cdot, t)}{\partial t}\right\|^2.
\]
Inserting (3.9)-(3.11) into (3.9) and integrating the obtained equality over \((0, \tau)\), it yields

\[
\alpha \|\zeta(n)(., \tau)\|^2 + \left\| \frac{\partial \zeta(n)(., \tau)}{\partial t} \right\|_{B^1_2(\Omega)}^2 + 2\beta \int_0^\tau \left\| \frac{\partial \zeta(n)(., t)}{\partial t} \right\|_{B^1_2(\Omega)}^2 \, dt = 2 \int_0^\tau \left( F^{(n-1)}(., t), \frac{\partial \zeta(n)(., t)}{\partial t} \right)_{B^1_2(\Omega)} \, dt.
\]  

(3.13)

According to the Cauchy inequality and the Lipschitz condition (3.7), it follows

\[
\alpha \|\zeta(n)(., \tau)\|^2 + \left\| \frac{\partial \zeta(n)(., \tau)}{\partial t} \right\|_{B^1_2(\Omega)}^2 + 2\beta \int_0^\tau \left\| \frac{\partial \zeta(n)(., t)}{\partial t} \right\|_{B^1_2(\Omega)}^2 \, dt \leq 2\varepsilon L^2 \left( \|\zeta^{(n-1)}\|_{L^2(I, B^1_2(\Omega))}^2 + \left\| \frac{\partial \zeta^{(n-1)}}{\partial t} \right\|_{L^2(I, B^1_2(\Omega))}^2 \right) + \frac{1}{\varepsilon} \int_0^\tau \left\| \frac{\partial \zeta(n)(., t)}{\partial t} \right\|_{B^1_2(\Omega)}^2 \, dt.
\]

(3.14)

Applying inequality (2.2) to the first term on the left-hand side of the last inequality and the last term on the right-hand side, and setting \(\varepsilon = \frac{\alpha}{4\beta}\), we get

\[
\frac{2\alpha}{L^2} \|\zeta(n)(., \tau)\|_{B^1_2(\Omega)}^2 + \left\| \frac{\partial \zeta(n)(., \tau)}{\partial t} \right\|_{B^1_2(\Omega)}^2 \leq \frac{L^2}{2\beta} \left( \|\zeta^{(n-1)}\|_{L^2(I, B^1_2(\Omega))}^2 + \left\| \frac{\partial \zeta^{(n-1)}}{\partial t} \right\|_{L^2(I, B^1_2(\Omega))}^2 \right).
\]

(3.14)

Integrating (3.14) over \(I\), we obtain

\[
\|\zeta(n)\|_{H^1(I, B^1_2(\Omega))}^2 \leq \frac{L^2}{2\beta \min (L^2, 2\alpha)} \|\zeta^{(n-1)}\|_{H^1(I, B^1_2(\Omega))}^2.
\]

(3.15)

This gives (3.8). \(\blacksquare\)

4. Existence, uniqueness and continuous dependence of the solution

**Theorem 2.** Hypotheses as in Theorem 3.1, moreover it is assumed that

\[
L < \sqrt{2\beta \min (L^2, 2\alpha)}.
\]

(4.1)

Then, problem (1.10)-(1.12) admits a weak solution in the space \(H^1(I, B^1_2(\Omega))\).

**Proof.** It is easy to observe that if, in inequality (3.15), we have

\[
c_4 < 1, \quad \text{i.e., } L < \sqrt{2\beta \min (L^2, 2\alpha)},
\]

c_4 < 1, i.e., L < \sqrt{2\beta \min (L^2, 2\alpha)},
then the series $\sum_{n=1}^{\infty} \zeta^{(n)}$ converges. Therefore, the sequence

$$z^{(n)}(x, t) = \sum_{m=0}^{n-1} (z^{(m+1)}(x, t) - z^{(m)}(x, t)) + z^{(0)}(x, t), \quad n = 1, 2, \ldots$$

converges to $\bar{z}(x, t)$ in $H^1(I, B^1_2(\Omega))$. Thus, it remains to establish the following Lemma: \Box

**Lemma 3.** The limit function $\bar{z}(x, t)$ is the solution of problem (3.4)-(3.6), in the sense of Definition 2.1.

**Proof.** We must show that $\bar{z}$ satisfies identity (2.3). For this, we take the weak formulation of the approximated problem

$$A (z^{(n)}, v) = \left( f^{(n)} \left( \ldots, z^{(n)}, \frac{\partial z^{(n)}}{\partial t} \right), v \right)_{L^2(I, B^1_2(\Omega))}$$

$$+ \beta \left( \varphi^{(n)}, v(., 0) \right) + \left( \psi^{(n)}, v(., 0) \right)_{B^1_2(\Omega)}. \tag{4.2}$$

The last identity imply that, when $n \to \infty$, the following holds, for all $v \in V$ and $n = 1, 2, \ldots$:

$$\lim_{n \to \infty} A (z^{(n)} - \bar{z}, v) + A (\bar{z}, v)$$

$$= \lim_{n \to \infty} \left( f^{(n)} \left( \ldots, z^{(n)}, \frac{\partial z^{(n)}}{\partial t} \right) - f \left( \ldots, z, \frac{\partial z}{\partial t} \right), v \right)_{L^2(I, B^1_2(\Omega))}$$

$$+ \left( f \left( \ldots, z, \frac{\partial z}{\partial t} \right), v \right)_{L^2(I, B^1_2(\Omega))}$$

$$+ \beta \lim_{n \to \infty} \left( \varphi^{(n)} - \varphi, v(., 0) \right) + \beta \left( \varphi, v(., 0) \right)$$

$$+ \lim_{n \to \infty} \left( \psi^{(n)} - \psi, v(., 0) \right)_{B^1_2(\Omega)} + \left( \psi, v(., 0) \right)_{B^1_2(\Omega)}. \tag{4.3}$$

According to Schwarz inequality and inequality (2.2), we get

$$A (z^{(n)} - \bar{z}, v)$$

$$\leq \max \left( \frac{1}{\sqrt{2}}, \alpha + \beta \right) \left( \| z^{(n)} - \bar{z} \|_{L^2(I, L^2_0(\Omega))} \right.$$

$$\left. + \left\| \frac{\partial z^{(n)}}{\partial t} - \frac{\partial \bar{z}}{\partial t} \right\|_{L^2(I, B^1_2(\Omega))} \right) \| v \|_{H^1(I, L^2_0(\Omega))}. \tag{4.4}$$

By taking into account of Corollary 4.2 in [6], we have

$$z^{(n)} \to \bar{z} \quad \text{in} \ L^2 \left( I, L^2_0(\Omega) \right), \tag{4.5}$$

$$\frac{\partial z^{(n)}}{\partial t} \to \frac{\partial \bar{z}}{\partial t} \quad \text{in} \ L^2 \left( I, B^1_2(\Omega) \right), \tag{4.6}$$
and when we pass to the limit in (4.4) as \( n \to \infty \), we obtain

\[
\lim_{n \to \infty} A \left( z^{(n)} - \overline{z}, v \right) = 0.
\]

According to inequality (3.7), (4.5) and (4.6), we get

\[
\left( f^{(n)} \left( \ldots, z^{(n)}, \frac{\partial z^{(n)}}{\partial t} \right) - f \left( \ldots, z, \frac{\partial z}{\partial t} \right), v \right)_{L^2(I,B^1_1(\Omega))} \\
\leq L \left( \| z^{(n)} - \overline{z} \|_{L^2(I,B^1_1(\Omega))} + \left\| \frac{\partial z^{(n)}}{\partial t} - \frac{\partial \overline{z}}{\partial t} \right\|_{L^2(I,B^1_1(\Omega))} \right) \| v \|_{L^2(I,B^1_2(\Omega))} \\
\leq L \left( \frac{l^2}{2} \| z^{(n)} - \overline{z} \|_{L^2(I,L^2_2(\Omega))} + \left\| \frac{\partial z^{(n)}}{\partial t} - \frac{\partial \overline{z}}{\partial t} \right\|_{L^2(I,B^1_2(\Omega))} \right) \| v \|_{L^2(I,L^2_2(\Omega))}.
\]

If we pass to the limit in (4.8) as \( n \to \infty \), we have

\[
\lim_{n \to \infty} \left( f^{(n)} \left( \ldots, z^{(n)}, \frac{\partial z^{(n)}}{\partial t} \right) - f \left( \ldots, z, \frac{\partial z}{\partial t} \right), v \right)_{L^2(I,B^1_2(\Omega))} = 0.
\]

It then follows that the limit function \( \overline{z}(x,t) \) satisfies identity (2.3) almost everywhere in \( Q \).

Thus we deduce that the limit-function \( \overline{z}(x,t) \) is a weak solution of problem (1.4)-(1.6), in the sense of Definition 2.1.

**Theorem 4.** Under hypotheses of Theorem 4.1 the solution of problem (1.10)-(1.12) is unique.

**Proof.** Let \( u_1(x,t) \) and \( u_2(x,t) \) be two solutions of the same quasilinear problem (3.4)-(3.6), then it is for the difference

\[ z(x,t) = u_1(x,t) - u_2(x,t), \]

we obtain

\[
\frac{\partial^2 z}{\partial t^2} - \alpha \frac{\partial^2 z}{\partial x^2} - \alpha \frac{\partial^3 z}{\partial t \partial x^2} = F(x,t),
\]

\[
z(x,0) = 0, \quad \frac{\partial z(x,0)}{\partial t} = 0,
\]

\[
\frac{\partial z(0,t)}{\partial x} = 0, \quad \int_\Omega z(x,t)dx = 0.
\]

where

\[ F(x,t) = f \left( x,t, u_1, \frac{\partial u_1}{\partial t} \right) - f \left( x,t, u_2, \frac{\partial u_2}{\partial t} \right). \]

We proceed as in the proof of Theorem 3.1, we get

\[ \| z \|_{H^1(I,B^1_2(\Omega))} \leq c_4 \| z \|_{H^1(I,B^1_2(\Omega))}; \]
Since \( c_4 < 1 \), then
\[
\|z\|_{H^1(I,B^1_2(\Omega))} = \|u_1 - u_2\|_{H^1(I,B^1_2(\Omega))} = 0,
\]
from which we have the:

**Theorem 5.** If \( u(x,t) \) and \( u^*(x,t) \) are two solutions of problem (1.4)-(1.6) corresponding to \((\varphi, \psi, f)\) and \((\varphi^*, \psi^*, f^*)\) respectively, and if there exist a continuous nonnegative function \( K(t) \) and a positive constant \( L \) such that the following estimate
\[
(4.13) \quad \|f(\cdot, t, p^1, q^1) - f^*(\cdot, t, p^2, q^2)\|_{B^1_2(\Omega)} \\
\leq K(t) + L \left( \|p^1 - p^2\|_{B^1_2(\Omega)} + \|q^1 - q^2\|_{B^1_2(\Omega)} \right)
\]
fulfils for all \( u, u^* \in B^1_2(\Omega) \) and all \( t \in I \), then
\[
(4.14) \quad \|u(., s) - u^*(., s)\|_{B^1_2(\Omega)} \\
\leq c_6 \left( \|\varphi - \varphi^*\|^2 + \|\psi - \psi^*\|^2_{B^2_2(\Omega)} + \int_0^s K^2(t) dt \right),
\]
where
\[
c_6 = \frac{\max(4/L, (1 + 4L) l^2/2 + 2\beta^2/\alpha)}{\min(1, L/4)} \exp\left( \frac{Ls}{\min(L, 4)} \right).
\]

**Proof.** Let
\[
\rho(x,t) = u(x,t) - u^*(x,t),
\]
then \( \rho(x,t) \) verifies
\[
(4.15) \quad \frac{\partial^2 \rho}{\partial t^2} - \alpha \frac{\partial^2 \rho}{\partial x^2} - \beta \frac{\partial^3 \rho}{\partial t \partial x^2} = f \left( x, t, u, \frac{\partial u}{\partial t} \right) - f^* \left( x, t, u^*, \frac{\partial u^*}{\partial t} \right),
\]
\[
(4.16) \quad \rho(x,0) = \varphi(x) - \varphi^*(x) = \rho_0(x),
\]
\[
\frac{\partial \rho(x,0)}{\partial t} = \psi(x) - \psi^*(x) = \rho_1(x),
\]
\[
(4.17) \quad \frac{\partial \rho(0,t)}{\partial x} = 0, \quad \int_\Omega \rho(x,t) dx = 0.
\]
Considering the weak formulation of problem (4.7)-(4.9), we have
\[
(4.18) \quad -\left( \frac{\partial \rho}{\partial t}, \frac{\partial v}{\partial t} \right)_{L^2(I,B^1_2(\Omega))} + \alpha (\rho, v)_{L^2(I,L^2_0(\Omega))} - \beta \left( \rho, \frac{\partial v}{\partial t} \right)_{L^2(I,L^2_0(\Omega))} \\
= \left( f(\cdot, t, u, \frac{\partial u}{\partial t}) - f^*(\cdot, t, u^*, \frac{\partial u^*}{\partial t}), v \right)_{L^2(I,B^1_2(\Omega))} \\
+ \beta (\rho_0, v(., 0)) + (\rho_1, v(., 0))_{B^1_2(\Omega)}.
\]
Taking
\[ v = \begin{cases} 
0, & \text{for } s \leq t \leq T, \\
\int_s^t \rho(x, \tau) \, d\tau, & \text{for } 0 \leq t \leq s,
\end{cases} \tag{4.19} \]

where \( s \) is an arbitrary fixed number in \([0, T]\).

Substituting \( v \) from (4.11) into (4.10), and express \( \rho \) and \( \frac{\partial \rho}{\partial t} \) in terms of \( v \) and its derivatives, we get after changing signs in the obtained equality
\[
\int_0^s \left( \frac{\partial^2 v (\cdot, t)}{\partial t^2}, \frac{\partial v (\cdot, t)}{\partial t} \right)_{B^2_1(\Omega)} \, dt - \alpha \int_0^s \left( \frac{\partial v (\cdot, t)}{\partial t}, v (\cdot, t) \right)_{B^2_1(\Omega)} \, dt \\
+ \beta \int_0^s \left\| \frac{\partial v (\cdot, t)}{\partial t} \right\|^2_{B^2_1(\Omega)} \, dt \\
= - \int_0^s \left( f (\cdot, t, u (\cdot, t), \frac{\partial u (\cdot, t)}{\partial t}) - f^* (\cdot, t, u^* (\cdot, t), \frac{\partial u^* (\cdot, t)}{\partial t}), v (\cdot, t) \right)_{B^2_1(\Omega)} \, dt \\
- \beta (\rho_0, v (\cdot, 0)) - (\rho_1, v (\cdot, 0))_{B^2_1(\Omega)}.
\]

If we integrate the first two terms on the left-hand side of the last equality, seeing that
\[ \frac{\partial v}{\partial t} \bigg|_{t=0} = \rho_0 (x) \]
and
\[ v|_{t=s} = 0, \]
and we apply condition (4.5), we obtain
\[
\left\| \frac{\partial v (\cdot, s)}{\partial t} \right\|_{B^2_1(\Omega)}^2 + \alpha \left\| v (\cdot, 0) \right\|^2 + 2\beta \int_0^s \left\| \frac{\partial v (\cdot, t)}{\partial t} \right\|^2_{B^2_1(\Omega)} \, dt \\
\leq 2 \int_0^s \left( K (t) \left\| v (\cdot, t) \right\|_{B^2_1(\Omega)} + L \left( \left\| \frac{\partial v (\cdot, t)}{\partial t} \right\|_{B^2_1(\Omega)} \right) \left\| v (\cdot, t) \right\|_{B^2_1(\Omega)} \\
+ \left\| \frac{\partial^2 v (\cdot, t)}{\partial t^2} \right\|_{B^2_1(\Omega)} \left\| v (\cdot, t) \right\|_{B^2_1(\Omega)} \right) \, dt \\
+ \left\| \rho_0 \right\|^2_{B^2_1(\Omega)} - 2\beta (\rho_0, v (\cdot, 0)) - 2 (\rho_1, v (\cdot, 0))_{B^2_1(\Omega)} \\
\leq 2 \int_0^s K (t) \left\| v (\cdot, t) \right\|_{B^2_1(\Omega)} \, dt - L \left\| v (\cdot, 0) \right\|^2_{B^2_1(\Omega)} \\
- 2L \left\| \rho_0 \right\|_{B^2_1(\Omega)} \left\| v (\cdot, 0) \right\|_{B^2_1(\Omega)} - 2L \int_0^s \left\| \frac{\partial v (\cdot, t)}{\partial t} \right\|^2_{B^2_1(\Omega)} \, dt + \left\| \rho_0 \right\|^2_{B^2_1(\Omega)} \\
+ 2\beta \left\| \rho_0 \right\| \left\| v (\cdot, 0) \right\| + 2 \left\| \rho_1 \right\|_{B^2_1(\Omega)} \left\| v (\cdot, 0) \right\|_{B^2_1(\Omega)}.
\]
from which we have

\[
(4.20) \quad \left\| \frac{\partial v(., s)}{\partial t} \right\|_{B^1_2(\Omega)}^2 + \frac{\alpha}{2} \left\| v(., 0) \right\|^2 + \frac{L}{2} \left\| v(., 0) \right\|_{B^1_2(\Omega)}^2 \\
\leq \frac{8}{L} \int_0^s K^2(t) \, dt + \frac{L}{8} \int_0^s \left\| v(., t) \right\|_{B^1_2(\Omega)}^2 \, dt \\
+ \left( 1 + 4L \right) \left( \frac{l^2}{2} + \frac{2\beta^2}{\alpha} \right) \left\| \rho_0 \right\|^2 + \frac{4}{L} \left\| \rho_1 \right\|_{B^1_2(\Omega)}^2.
\]

Let us introduce a new function

\[
\delta(x, t) := \int_t^0 \rho(x, \tau) \, d\tau.
\]

Using (4.11), it follows

\[
v(x, t) = \delta(x, s) - \delta(x, t)
\]

and

\[
v(x, 0) = \delta(x, s).
\]

In view of this, it yields from (4.12) that

\[
(4.21) \quad \left\| \frac{\partial v(., s)}{\partial t} \right\|_{B^1_2(\Omega)}^2 + \frac{\alpha}{2} \left\| \delta(., s) \right\|^2 + \frac{L}{4} \left\| \delta(., s) \right\|_{B^1_2(\Omega)}^2 \\
\leq \frac{8}{L} \int_0^s K^2(t) \, dt + \frac{L}{4} \int_0^s \left\| \delta(., t) \right\|_{B^1_2(\Omega)}^2 \, dt \\
+ \left( 1 + 4L \right) \left( \frac{l^2}{2} + \frac{2\beta^2}{\alpha} \right) \left\| \rho_0 \right\|^2 + \frac{4}{L} \left\| \rho_1 \right\|_{B^1_2(\Omega)}^2,
\]

therefore, we obtain by omitting the second term on the left-hand side of (4.13)

\[
\left\| \frac{\partial v(., s)}{\partial t} \right\|_{B^1_2(\Omega)}^2 + \left\| \delta(., s) \right\|_{B^1_2(\Omega)}^2 \\
\leq c_5 \left( \int_0^s K^2(t) \, dt + \left\| \rho_0 \right\|^2 + \left\| \rho_1 \right\|_{B^1_2(\Omega)}^2 \right) \\
+ \frac{L}{\min(L, 4)} \int_0^s \left( \left\| \frac{\partial v(., t)}{\partial t} \right\|_{B^1_2(\Omega)}^2 + \left\| \delta(., t) \right\|_{B^1_2(\Omega)}^2 \right) \, dt,
\]

where

\[
c_5 = \frac{\max \left( \frac{4}{L}, (1 + 4L) \frac{l^2}{2} + \frac{2\beta^2}{\alpha} \right)}{\min \left( 1, \frac{L}{4} \right)}.
\]
It then follows by Gronwall’s Lemma [3]

\[
\left\| \frac{\partial v}{\partial t} \right\|_{B^1_2(\Omega)}^2 + \left\| \delta (., s) \right\|_{B^1_2(\Omega)}^2 \leq c_5 \exp \left( \frac{Ls}{\min(L, 4)} \right) \left( \int_0^s K^2(t) \, dt + \left\| \rho_0 \right\|_{B^1_2(\Omega)}^2 + \left\| \rho_1 \right\|_{B^1_2(\Omega)}^2 \right).
\]

Hence

\[
\left\| u (., s) - u^* (., s) \right\|_{B^1_2(\Omega)}^2 \leq c_6 \left( \int_0^s K^2(t) \, dt + \left\| \varphi - \varphi^* \right\|_{B^1_2(\Omega)}^2 + \left\| \psi - \psi^* \right\|_{B^1_2(\Omega)}^2 \right),
\]

where

\[
c_6 = c_5 \exp \left( \frac{Ls}{\min(L, 4)} \right).
\]

This completes the proof of Theorem 4.4. ⊡

References


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