Inequalities for Powers of the Numerical Radii of Hilbert Space Operators

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Abstract

Several inequalities involving powers of the numerical radii of Hilbert space operators, which generalize earlier inequalities, are given. These inequalities are based on the generalized mixed Schwarz inequality, operator inequalities and some classical inequalities for nonnegative real numbers.

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1 Introduction

Let $B(H)$ denotes the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $H$ with inner product $\langle ., . \rangle$, here $\langle ., . \rangle$ denotes the standard inner product in $\mathbb{C}^n$. For $A \in B(H)$, the usual operator norm of an operator $A$, denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{\|x\|=1} \|Ax\|, \text{ for all } x \in H,$$

where $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$. A unitarily invariant norm $\|\| \cdot \||$ on $H$ is a norm on the ideal $C_{\|\|}$ of $B(H)$, making $C_{\|\|}$ a Banach space and satisfying $\|UA\| = \cdots$
\[ \| | A | \| \text{ for all } A \in B(H) \text{ and all unitary operators } U \text{ and } V \in B(H). \] It is called weakly unitarily invariant norm if \[ \| | UA U^* | \| = \| | A | \| \] for all \( A \in B(H) \) and all unitary operator \( U \in B(H) \).

The most familiar example of weakly unitarily (but not unitarily) invariant norm is the numerical radius \( w(A) \) defined by
\[
w(A) = \sup \{ |\lambda| : \lambda \in W(A) \},
\]
where \( W(A) \) is the numerical range of \( A \) defined by
\[
W(A) = \{ \langle Ax, x \rangle : x \in H, \| x \| = 1 \}.
\]
It is well known that for every \( A \in B(H) \)
\[
\frac{1}{2} \| A \| \leq w(A) \leq \| A \|. \tag{1}
\]

For basic properties of the numerical radius, we refer to [1], [3], and [4]. The inequalities in (1) have been improved considerably by Kittaneh in [8] and [9]. It has been shown that if \( A \in B(H) \), then
\[
w(A) \leq \frac{1}{2} \left( \| A \| + \| A^2 \|^{\frac{1}{2}} \right) \tag{2}
\]
and
\[
\frac{1}{4} \| A^* A + A A^* \| \leq w^2(A) \leq \frac{1}{2} \| A^* A + A A^* \|. \tag{3}
\]
Recently, El-Haddad and Kittaneh [5] generalized some inequalities for powers of the numerical radii. It has been shown that if \( A \in B(H) \), then for \( 0 < \alpha < 1 \) and \( r \geq 1 \), we have
\[
w^r(A) \leq \frac{1}{2} \| | A |^{2r\alpha} + | A^* |^{2r(1-\alpha)} | \| \tag{4}
\]
and
\[
w^{2r}(A) \leq \| \alpha | A |^{2r} + (1 - \alpha) | A^* |^{2r} \| . \tag{5}
\]
Other recent numerical radius inequalities have been obtained by Dragomir [2], which are related to the Euclidean radius of two Hilbert space operators. Also, it has been proved in [5] that if \( A, B \in B(H) \) and \( r \geq 1 \), then
\[
w^r(A + B) \leq 2^{r-2} \| | A |^{2r\alpha} + | A^* |^{2r(1-\alpha)} + | B |^{2r\alpha} + | B^* |^{2r(1-\alpha)} \| . \tag{6}
\]
In particular, if \( A = B \), then inequality (6) reduces to inequality (4).

In the next section of this paper, we generalize inequalities (2), (4), and (5) using some classical inequalities for nonnegative real numbers and some operator inequalities.
2 Numerical radius inequalities

In this section, we prove useful numerical radius inequalities for Hilbert space operators. To prove our generalized numerical radius inequalities, we need several well-known lemmas. The first lemma follows from spectral theorem for positive operators and Jensen’s inequality, see [6].

**Lemma 2.1** Let $A$ in $B(H)$ be positive and let $x$ in $H$ be any unit vector. Then
\[
\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle \text{ for all } r \geq 1 \tag{7}
\]
and
\[
\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r \text{ for all } 0 \leq r \leq 1. \tag{8}
\]

The second lemma is a generalized form of the mixed Schwarz inequality which has been proved by Kittaneh [6].

**Lemma 2.2** Let $T$ be an operator in $B(H)$ and let $f$ and $g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then
\[
|\langle Tx, y \rangle| \leq \|f(|T|) x\| \|g(|T^*|) y\| \text{ for all } x, y \text{ in } H. \tag{9}
\]

The third lemma is a simple consequence of the classical Jensen’s inequality concerning the convexity of the function $f(t) = t^r$, $r \geq 1$.

**Lemma 2.3** If $a$ and $b$ are nonnegative real numbers, then
\[
(a + b)^r \leq 2^{r-1} (a^r + b^r) \text{ for } r \geq 1. \tag{10}
\]

The fourth lemma was given by Kittaneh [7].

**Lemma 2.4** If $A$ and $B$ are positive operators in $B(H)$, then
\[
\|A + B\| \leq \max (\|A\| + \|B\|) + \left\| A^{\frac{1}{2}} B^{\frac{1}{2}} \right\|. \tag{11}
\]

The fifth lemma contains usual operator norm inequalities for positive operators, which can be found in [1].

**Lemma 2.5** Let $A$ and $B$ be positive operators in $B(H)$. Then
\[
\|AB\|^r \leq \|A^r B^r\| \text{ for } r \geq 1 \tag{12}
\]

and
\[
\|A^r B^r\| \leq \|AB\|^r \text{ for } 0 < r \leq 1. \tag{13}
\]
Our first result is a generalization of inequality (4).

**Theorem 2.6** Let $A$ be any operator in $B(H)$, and let $f$ and $g$ be as in Lemma 2.1. Then for $r \geq 1$, we have

$$w^r(A) \leq \frac{1}{2} \| f^{2r}(\|A\|) + g^{2r}(\|A^*\|) \|.$$  \hspace{1cm} (14)

**Proof:** For every unit vector $x \in H$, we have

\[
|\langle Ax, x \rangle |^{2r} \leq \langle f^2(\|A\|) x, x \rangle^r \langle g^2(\|A^*\|) x, x \rangle^r \quad \text{(by Lemma 2.2)}
\]

\[
\leq \langle f^{2r}(\|A\|) x, x \rangle \langle g^{2r}(\|A^*\|) x, x \rangle \quad \text{(by Lemma 2.1)}
\]

\[
\leq \frac{1}{4} \left( \langle f^{2r}(\|A\|) x, x \rangle + \langle g^{2r}(\|A^*\|) x, x \rangle \right)^2
\]

(by the arithmetic-geometric mean inequality)

Thus,

$$|\langle Ax, x \rangle |^r \leq \frac{1}{2} \left( \langle f^{2r}(\|A\|) + g^{2r}(\|A^*\|) \rangle x, x \right),$$

and so

$$\sup \{|\langle Ax, x \rangle |^r : x \in H, \|x\| = 1 \} \leq \frac{1}{2} \sup \{ (\langle f^{2r}(\|A\|) + g^{2r}(\|A^*\|) \rangle x, x) : x \in H, \|x\| = 1 \}.$$ 

Hence,

$$w^r(A) \leq \frac{1}{2} \| f^{2r}(\|A\|) + g^{2r}(\|A^*\|) \|.$$ 

Our second result is related to inequality (6) and generalizes inequality (5), which gives a numerical radius inequality for sum of two operators.

**Theorem 2.7** Let $A$ and $B$ be operators in $B(H)$, and let $\alpha$ be any positive real number such that $0 < \alpha < 1$. Then, for $r \geq 2$,

$$w^r(A + B) \leq 2^{r-1} \| \alpha (\|A\|^{r} + \|B\|^{r}) + (1 - \alpha) (\|A^*\|^{r} + \|B^*\|^{r}) \|.$$  \hspace{1cm} (15)

**Proof:** For every unit vector $x \in H$, we have

$$\|((A + B)x, x)\|^r = |\langle Ax, x \rangle + \langle Bx, x \rangle |^r$$

\[
\leq (|\langle Ax, x \rangle| + |\langle Bx, x \rangle|)^r \quad \text{(by the triangle inequality)}
\]

\[
\leq 2^{r-1} (|\langle Ax, x \rangle|^r + |\langle Bx, x \rangle|^r) \quad \text{(by Lemma 2.3)}
\]

\[
\leq 2^{r-1} \left( \langle |A|^{2\alpha} x, x \rangle^\frac{r}{2} \langle |A^*|^{2(1-\alpha)} x, x \rangle^\frac{r}{2} + \langle |B|^{2\alpha} x, x \rangle^\frac{r}{2} \langle |B^*|^{2(1-\alpha)} x, x \rangle^\frac{r}{2} \right)
\]

(by the mixed Cauchy-Schwarz inequality)

\[
\leq 2^{r-1} \left( (\|A\|^{r})^\alpha \langle |A^*|^{r} x, x \rangle^{(1-\alpha)} + (\|B\|^{r})^\alpha \langle |B^*|^{r} x, x \rangle^{(1-\alpha)} \right)
\]

(by Lemma 2.1)

\[
\leq 2^{r-1} (\alpha |\langle A\|^{r} x, x \rangle + (1 - \alpha) |\langle A^*\|^{r} x, x \rangle + \alpha |\langle B\|^{r} x, x \rangle + (1 - \alpha) |\langle B^*\|^{r} x, x \rangle)
\]

(by the Young’s inequality)

\[
= 2^{r-1} \langle \alpha (|A|^{r} + |B|^{r}) + (1 - \alpha) (|A^*|^{r} + |B^*|^{r}) \rangle x, x \rangle.
\]
Thus,
\[ |\langle (A + B) x, x \rangle|^r \leq 2^{r-1} \left( \alpha (|A|^r + |B|^r) + (1 - \alpha) (|A^*|^r + |B^*|^r) \right) x, x \],
so,
\[ \omega^r (A + B) \leq 2^{r-1} \| \alpha (|A|^r + |B|^r) + (1 - \alpha) (|A^*|^r + |B^*|^r) \| . \]

Using similar procedures as in the proof of Theorem 2.6, one can get the following operator norm inequality.

**Theorem 2.8** Let \( A \) and \( B \) be operators in \( B(H) \), and let \( \alpha \) be any positive real number such that \( 0 < \alpha < 1 \). Then, for \( r \geq 2 \),
\[ \| A + B \|^r \leq 2^{r-1} (\alpha \| |A|^r + |B|^r \| + (1 - \alpha) \| |A^*|^r + |B^*|^r \| ) . \] (16)

The next result is a generalization of inequality (2).

**Theorem 2.9** Let \( A \) be any operator in \( B(H) \) and \( r \geq 1 \). Then
\[ \omega^r (A) \leq \frac{1}{2} \left( \| |A|^r + \| A^r |A^*|^r \|^{\frac{1}{2}} \right) . \] (17)

In particular,
\[ \omega^r (A) \leq \frac{1}{2} \left( \| |A|^r + \| A^2 \|^{\frac{1}{2}} \right) \]
for \( 1 \leq r \leq 2 \).

**Proof:** Using inequality (4) for \( \alpha = \frac{1}{2} \), we have
\[
\omega^r (A) \leq \frac{1}{2} \| |A|^r + |A^*|^r \| \\
\leq \frac{1}{2} \max (\| |A|^r \| , \| |A^*|^r \| ) + \left\| |A^\frac{r}{2} |A^*|^\frac{r}{2} \right\| \quad \text{(by Lemma 2.4)} \\
\leq \frac{1}{2} \left( \| |A|^r + \| |A|^r |A^*|^r \|^{\frac{1}{2}} \right) . \quad \text{(by Lemma 2.5)}
\]

For the particular case apply Lemma 2.5, to get
\[
\left\| |A^\frac{r}{2} |A^*|^\frac{r}{2} \right\| \leq \| |A| \| |A^*| \|^{\frac{1}{2}} = \| A^2 \|^{\frac{r}{2}}, \quad \text{for } 1 \leq r \leq 2 .
\]

In the last equality, we can use the polar decomposition or the basic properties of the singular values.
References


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