Pointwise Convergence of Multiwavelet Packet Series

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Abstract

This paper deals with the pointwise convergence of multiwavelet packet expansions associated with dilatim matrix A. It have been shown that such expansion converges uniformly on compact subsets.

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1. Introduction

Multiwavelets are a natural extension and generalization of traditional wavelets. They have received considerable attention from the wavelet research communities both in the theory as well as in applications. They can be seen as vector valued wavelets that satisfy conditions in which matrices are involved, rather than scalars, as in the wavelet case. Multiwavelets can own symmetry, orthogonality, short support and high order vanishing moments, however traditional wavelets can not possess all these properties at the same time. Multiwavelet system provides perfect reconstruction while preserving length, good performance at boundaries and high order of approximation.
The problem of convergence of the wavelet series has been studied by Meyer [9], Walter [12] and Kelly et al. [7, 8]. Meyer was amongst the first to study convergence results for wavelet expansions. He has shown that the regular wavelet expansions converge in $L^p$, $1 \leq p < \infty$ and also in $L^\infty$ for expansions of uniformly continuous functions, the expansion of continuous functions converge everywhere. The results in [9] were based on the assumption of so called regularity for the basic wavelets and their derivatives. In addition, Walter [9] established pointwise convergence results for regular wavelet expansions of continuous functions. Kelly et al. [7, 8] have extended and obtained results analogous to those obtained by Carleson [4] and Hunt [5] for the Fourier series. In contrast, the results in [7, 8] assumed only that the wavelets being used be bounded by radial decreasing $L^1$-functions. In [8], it is shown that the wavelet expansions of a function belonging to $L^p$ converges pointwise everywhere on the Lebesgue set of a given function, for $1 \leq p < \infty$. On the other hand Tao [11] has extended the results of Meyer [9] and Kelly et al. [7, 8] and has shown that the wavelet expansion of any $L^p$-function converges pointwise almost everywhere under the wavelet projections, hard sampling and soft sampling summation methods, for $1 < p < \infty$.

Shen and Tan [10] have proposed some new thresholding operators for studying the pointwise convergence results for multiwavelets with dilation factor 2 which generalizes the results of Kelly et al. [7, 8] and Tao [11]. Recently, Ahmad and Shah have studied pointwise convergence results for multiwavelets in [1] and have shown that such expansions of $L^p(\mathbb{R})$ functions ($1 \leq p \leq \infty$) converges pointwise almost everywhere on the Lebesgue set of the functions being expanded. Motivated and inspired by the importance of multiwavelet packets, in the present paper, we study the pointwise convergence of multiwavelet packet expansions associated with dilation matrix.

2. Preliminaries

Assume that we have a dilation matrix $A$ preserving $\mathbb{Z}^d$, i.e, $A$ is a $d \times d$ matrix such that

(i) $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$

(ii) all eigen values $\lambda$ of $A$ satisfy $|\lambda| > 1$.

Property (i) implies that $A$ has integer entries and hence $|\det A|$ is an integer, and (ii) says that $|\det A|$ is greater than 1. Let $a = |\det A|$. Considering $\mathbb{Z}^d$ as an additive group, we see that $AZ^d$ is a normal subgroup of $\mathbb{Z}^d$ so we can form the cosets of $AZ^d$ in $\mathbb{Z}^d$. It is well known fact that the number of distinct cosets of $AZ^d$ in $\mathbb{Z}^d$ is equal to $a = |\det A|$ (see [13]).
Definition 2.1 [3, 6]. A sequence \( \{V_j : j \in \mathbb{Z}\} \) of closed subspaces of \( L^2(\mathbb{R}^d) \) is called a multiresolution analysis (MRA) of \( L^2(\mathbb{R}^d) \) of multiplicity \( L \) associated with the dilation matrix \( A \) if the following conditions are satisfied:

\[
\begin{align*}
(2.1) & \quad V_j \subset V_{j+1} \text{ for all } j \in \mathbb{Z}; \\
(2.2) & \quad f(x) \in V_j \iff f(Ax) \in V_{j+1}, \quad \forall j \in \mathbb{Z}, \, x \in \mathbb{R}^d; \\
(2.3) & \quad \bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \quad \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^d); \\
(2.4) & \quad \text{There exist } L \text{ functions } \varphi_1, \ldots, \varphi_L \in V_0, \text{ such that the system of functions } \{\varphi_\ell(x-k) : k \in \mathbb{Z}^d, 1 \leq \ell \leq L\} \text{ forms an orthonormal basis of } V_0. 
\end{align*}
\]

The \( L \) functions whose existence is asserted in (2.4) are called scaling functions of the given MRA.

For basic construction of multiwavelets and multiwavelet packets associated with multiresolution analysis of multiplicity \( L \) over dilation matrix \( A \), we refer to [2, 3, 6].

Definition 2.2 [2]. Let \( \{\omega^n_\ell : n \geq 0, 1 \leq \ell \leq L\} \) be the basic multiwavelet packets. The collection

\[
\mathcal{P} = \left\{\left|\text{det}A\right|^{j/2}\omega^n_\ell(A^{-1}x-k) : 1 \leq \ell \leq L, j \in \mathbb{Z}, k \in \mathbb{Z}^d\right\}
\]

is called the “general multiwavelet packets” associated with MRA \( \{V_j : j \in \mathbb{Z}\} \) of \( L^2(\mathbb{R}^d) \) of multiplicity \( L \) over dilation matrix \( A \).

Corresponding to some orthonormal scaling vector \( \Phi = \omega^0_\ell \), the family of multiwavelet packets \( \omega^n_\ell \) defines a family of subspaces of \( L^2(\mathbb{R}^d) \) as follows:

\[
(2.5) \quad U^n_j = \text{span}\left\{a^{j/2}\omega^n_\ell(A^jx-k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq \ell \leq L, \, n = 0, 1, \ldots\right\}.
\]

Observe that

\[
U^0_j = V_j, \quad U^1_j = W_j = \bigoplus_{r=1}^{a-1} U^r_j, \quad j \in \mathbb{Z}
\]

so that the orthogonal decomposition \( V_{j+1} = V_j \oplus W_j \), can be written as

\[
(2.6) \quad U^0_{j+1} = \bigoplus_{r=0}^{a-1} U^r_j.
\]
A generalization of this result for other values of \( n = 1, 2, \ldots \) can be written as

\[
U_{j+1}^n = \bigoplus_{r=0}^{a-1} U_{j+1}^{an+r}, \quad j \in \mathbb{Z}.
\]

**Lemma 2.3** [2]. If \( j \geq 0 \), then

\[
W_j = \bigoplus_{r=0}^{a-1} U_j^r = \bigoplus_{r=a}^{a^2-1} U_j^r = \bigoplus_{r=a^t}^{a^{t+1}-1} U_{j-t}^r, \quad t \leq j
\]

where \( U_j^n \) is defined in (2.5). Using this decomposition, we get the multiwavelet packets decomposition of subspaces \( W_j, j \geq 0 \).

If a function \( f \in L^2(\mathbb{R}^d) \), then

\[
f(x) \sim \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^L \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} C_{\ell,j,k}^n \omega_{\ell,j,k}^n(x)
\]

will be the multiwavelet packet expansion of \( f \) and \( C_{\ell,j,k}^n \) the multiwavelet packet coefficients, defined as

\[
C_{\ell,j,k}^n = \langle f, \omega_{\ell,j,k}^n(x) \rangle.
\]

Let \( P_j \) and \( Q_j \), respectively be the orthogonal projections onto the spaces \( V_j \) and \( W_j \) with the kernels \( P_j(x, y) \) and \( Q_j(x, y) \), defined as follows:

\[
P_j(x, y) = \sum_{k \in \mathbb{Z}^d} \Phi(x - k) \overline{\Phi(y - k)}
\]

and

\[
Q_j(x, y) = \sum_{r=1}^{a-1} \sum_{\ell=1}^L \sum_{k \in \mathbb{Z}^d} \psi_{\ell,j,k}^r(x) \overline{\psi_{\ell,j,k}^r(y)}.
\]

In the light of \( V_{j+1} = V_j \oplus W_j \), \( P_j(x, y) \) can be written as

\[
P_j(x, y) = \sum_{m \leq j} Q_m(x, y) = \sum_{r=1}^{a-1} \sum_{\ell=1}^L \sum_{m \leq j} \sum_{k \in \mathbb{Z}^d} \psi_{\ell,j,k}^r(x) \overline{\psi_{\ell,j,k}^r(y)}.
\]
Now, we consider a projection $Q_j^n$ onto $U_j^n$ with the kernel $Q_j^n(x, y)$ defined as

$$Q_j^n(x, y) = \sum_{\ell=1}^L \sum_{k \in \mathbb{Z}^d} \omega_{\ell,j,k}^n(x) \overline{\omega_{\ell,j,k}^n(y)},$$

where $\{\omega_{\ell,j,k}^n\}$ is a multiwavelet packet. Thus, we observe that $Q_j^0 = P_j$ and $Q_j^1 = Q_j$. In the light of equation (2.7), $Q_j$ can be expressed as

$$Q_j(x, y) = \sum_{n=a^j}^{a^{j+1}-1} Q_j^n(x, y)$$

$$= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^L \sum_{k \in \mathbb{Z}^d} \omega_{\ell,j,k}^n(x) \overline{\omega_{\ell,j,k}^n(y)}.$$ 

Thus (2.10) gives

$$P_j(x, y) = \sum_{m<j} Q_m(x, y) = \sum_{n=a^j}^{a^{j+1}-1} \sum_{m<j} Q_j^n(x, y)$$

$$= \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^L \sum_{m<j} \sum_{k \in \mathbb{Z}^d} \omega_{\ell,j,k}^n(x) \overline{\omega_{\ell,j,k}^n(y)}.$$ 

For a scaling function $\Phi$ associated with a dilation matrix $A$ considered above, the following results hold (see [13]):

$$\int_{\mathbb{R}^d} \Phi(x) \, dx = 1$$

$$\sum_{k \in \mathbb{Z}^d} \Phi(x - k) = 1$$

$$|\Phi(t)| \leq \frac{C_k}{(1 + |t|)^k}, \quad k = 1, 2, 3, ...$$

A sequence $\delta_m(x, y)$ of functions in $L^1(\mathbb{R}^d)$ is called a quasi-positive delta sequence if the following conditions are satisfied:

(i) there exists a constant $C$ such that

$$\int_{\mathbb{R}^d} |\delta_m(x, y)| \, dx \leq C, \quad \text{for all } y \in \mathbb{R}^d, m \in \mathbb{N}$$
(ii) there exists a vector \(c = (c_1, c_2, ..., c_d) > 0\) such that

\[
(2.18) \quad \int_{|y-c,y+c|} \delta_m(x, y) \, dx \to 1
\]

uniformly on compact subset of \(\mathbb{R}^d\) as \(m \to \infty\);

(iii) for each \(r > 0\)

\[
(2.19) \quad \sup_{|x-y| \geq r} |\delta_m(x, y)| \to 0 \quad \text{as} \quad m \to \infty.
\]

3. Convergence Results

We prove here a theorem on the pointwise convergence of two-dimensional multiwavelet packet expansions associated with the dilation matrix. The proof is also valid for higher dimensions. More precisely, we prove that a multiwavelet packet expansion associated with a dilation matrix of a continuous function \(f\) belonging to \(L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)\) converges uniformly on a compact subset.

The functions \(P_j f \in V_j\) is, in fact, the projection \(f\) onto \(V_j\). It can be written as

\[
(3.1) \quad P_j f(x) = \int_{\mathbb{R}^d} |\text{det}A|^{j/2} P_0(A^j x, A^j y) f(y) \, dy
\]

where

\[
P_0(x, y) = \sum_{k \in \mathbb{Z}^d} \Phi(x - k) \overline{\Phi(y - k)}.
\]

**Lemma 3.1.** The reproducing kernel \(P_j(x, y)\) of \(V_j\) as defined in (2.8), is a quasi-delta sequence.

**Proof.** For the sake of convenience we write the proof for the two-dimensional case. We have

\[
\int_{\mathbb{R}^2} |P_j(x, y)| \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\text{det}A|^{j/2} |P_j(A^j x, A^j y)| \, dx

= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |P_0(x, y)| \, dx

(3.2) \quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + |x - A^j y|)^{-k} \, dx = C \quad \text{by} \quad (2.16).
\]
We can write
\[
\int_{y-c}^{y+c} \int_{y-c}^{y+c} P_j(x, y) \, dx = \int_{A'(y_1-c_1)}^{A'(y_1+c_1)} \int_{A'(y_2-c_2)}^{A'(y_2+c_2)} P_0(x, A'y) \, dx
\]
\[
\leq \int_{t-A'y_1}^{t+A'y_1} \int_{t-A'y_2}^{t+A'y_2} P_0(x, t) \, dx
\]
\[
= \int_{t-A'y_1}^{t+A'y_1} \int_{t-A'y_2}^{t+A'y_2} - \int_{-\infty}^{t-A'y_1} \int_{-\infty}^{t-A'y_2} \, dx
\]
\[
= 1 - I_1 - I_2
\]

\[
I_1 \leq C \int_{t+A'y_1}^{\infty} \int_{t+A'y_2}^{\infty} \frac{1}{1 + (t - x)^k} \, dx
\]
\[
= C \int_{A'y_1}^{\infty} \int_{A'y_2}^{\infty} \frac{1}{1 + x^k} \, dx \rightarrow 0, \quad k > 1, \quad \text{as} \quad j \rightarrow \infty.
\]

Similarly, \( I_2 \rightarrow o \) as \( j \rightarrow \infty \). Hence (2.18) holds. Equation (2.19) can also be verified by using Equation (2.16).

Theorem 3.2. Let \( P_j(x, y) \) be a reproducing kernel associated with a dilation matrix \( A \), and let \( f \in L^1(\mathbb{R}^2) \) be continuous on an open set \( U \) in \( \mathbb{R}^2 \), then

\[
P_j f(x) = \int_{[y-\eta, y+\eta] \times \mathbb{R}} P_j(x, y) f(x) \, dx \rightarrow f(y)
\]
as \( j \rightarrow \infty \) uniformly on compact subsets of \( U \).

Proof. Let \( \eta > 0 \), then

\[
P_j f(x) = \int_{[y-\eta, y+\eta] \times \mathbb{R}} P_j(x, y) f(x) \, dx
\]
\[
+ \int_{[y+\eta, \infty] \times \mathbb{R}} P_j(x, y) f(x) \, dx
\]
\[
+ \int_{[-\infty, y+\eta] \times \mathbb{R}} P_j(x, y) f(x) \, dx
\]
\[
= f(y) \int_{[y-\eta, y+\eta] \times \mathbb{R}} P_j(x, y) \, dx
\]
\begin{align*}
+ \int_{[y-\eta,y+\eta] \times \mathbb{R}} P_j(x,y) \{ f(x) - f(y) \} \, dx \\
+ \int_{[y+\eta,\infty] \times \mathbb{R}} P_j(x,y) f(x) \, dx \\
+ \int_{(-\infty, y-\eta] \times \mathbb{R}} P_j(x,y) f(x) \, dx = I_1 + I_2 + I_3.
\end{align*}

(3.4)

Let $S$ be a compact subset of $U$, and let $Q$ be a closed subset contained in $U$ containing $S$. For any $y \in Q$, choose $\eta$ such that $0 < \eta < c$. Further, we restrict $\eta$ such that $|f(x) - f(y)| < \varepsilon$ for $y \in S$ and $|x - y| < \eta$. Therefore, from this it follows that

\begin{equation}
|I_2| \leq \varepsilon \int_{[y-\eta,y+\eta] \times \mathbb{R}} |P_j(x,y)| \, dx
\end{equation}

and

\begin{equation}
|I_3| \leq \sup_{\eta \leq |x-y|} |P_j(x,y)| \|f\|_{L^1(\mathbb{R}^2)} \text{ whenever } j \geq J_1
\end{equation}

where $J_1$ so large that

$$\sup_{\eta \leq |x-y|} |P_j(x,y)| < \varepsilon \text{ for } j \geq J_1.$$

We choose $J_2 \geq J_1$ so that

\begin{equation}
\left| 1 - \int_{[y-\eta,y+\eta] \times \mathbb{R}} P_j(x,y) \right| < \varepsilon \text{ whenever } j \geq J_2.
\end{equation}

(3.7)

The above inequality follows from Lemma 3.1 and Equation (2.18). By combining Equations (3.5), (3.6) and (3.7), we have

$$|f(y) - P_j f(y)| \leq |f(y) - I_1| + |I_2| + |I_3|$$

$$\leq |f(y)| \left| 1 - \int_{y-\gamma}^{y+\gamma} P_j(x,y) \, dx \right| + \varepsilon \int_{-\infty}^{\infty} |P_j(x,y)| \, dx + \varepsilon \|f\|_1$$

$$\leq \sup_{y \in [\alpha,\beta]} |f(y)| \varepsilon + \varepsilon C + \varepsilon \|f\|_1 \text{ for } j \geq J_2.$$
which gives desired uniform convergence convergence on \([\alpha, \beta]\) and hence on \(S\). This proves the theorem. □

**Corollary 3.3.** Let \(f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)\) be a continuous on a subset \(U\) and let \(P_j f\) be the projection of \(f\) into \(V_j\), then

\[
P_j f \to f \quad \text{as} \quad j \to \infty,
\]

uniformly on compact subsets of \(U\).

**Proof.** The proof of this corollary follows from Theorem 3.2 and Lemma 3.1. □

**References**

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