Category of Probabilistic Metric Spaces and a Fixed Point Theorem

Mohd. Rafi Segi Rahmat

School of Applied Mathematics
The University of Nottingham Malaysia Campus
Jalan Broga, 43500 Semenyih
Selangor Darul Ehsan, Malaysia
Mohd.Rafi@nottingham.edu.my

Mohd. Salmi Md. Noorani

School of Mathematical science
Universiti Kebangsaan Malaysia
43600 UKM, Bangi, Selangor Darul Ehsan, Malaysia

Abstract

In this paper, the probabilistic nonexpansive (PNE) mappings between probabilistic metric spaces were introduced and studied. The category of probabilistic metric spaces \((CPM^≈)\) is introduced and a fixed point theorem in \(CPM^≈\) is proved.

Mathematics Subject Classification: 18A99, 54E70, 47H10

Keywords: Category of Probabilistic Metric Space, fixed point theorem, probabilistic Cauchy tower, functor

1 Introduction

Menger [8] was introduced the notion of probabilistic metric (PM) space which is a generalization of the metric space. The study of this space was rapidly with the pioneering work of Schweizer and Skalar [12] and many others. Fixed point theorem in PM spaces was first initiated by Sehgal and Bharucha-Reid [13]. Recently, many authors [3, 4, 5, 6, 7, 9, 10, 11, 14] have contributed the work on fixed point theorem in PM spaces. Following the work of Alessi et al [1, 2], we introduce and study the notions of probabilistic nonexpansive (PNE), \(λ\)-adjoint and \(λ\)-isometry mappings between probabilistic metric spaces. We
also introduce the category of PM spaces \((CPM^\approx)\) and the \(B^\approx\) contraction of functors in \(CPM^\approx\) and obtained a fixed point theorem in \(CPM^\approx\).

All the notations and the notions are standard and follow the book [12].

A distance distribution function (briefly a d.d.f.) is a nondecreasing function from \(\mathbb{R}^+ = [0, +\infty]\) into \(\mathbb{R}\), which is left-continuous on \((0, \infty)\) and takes on the values \(F(0) = 0\) and \(F(+\infty) = 1\). The set of all d.d.f.s is denoted by \(\Delta^+\). The elements of \(\Delta^+\) are partially ordered via \(F \leq G\) if and only if \(F(t) \leq G(T)\) for all \(t \in \mathbb{R}^+\). The maximal element for \(\Delta^+\) in this order is the d.d.f. given by

\[
\varepsilon_0(t) = \begin{cases} 
0, & t = 0, \\
1, & t > 0.
\end{cases}
\]

Let \(F, G \in \Delta^+, h \in (0, 1]\) and let \([F, G; h]\) denote the condition \(G(t) \leq F(t + h) + h\) for all \(t \in (0, 1/h]\). The modified Levy metric (see page 45 of [12]) is the function \(d_L\) defined on \(\Delta^+ \times \Delta^+\) by

\[
d_L(F, G) = \inf \{h : \text{both } [F, G; h] \text{ and } [G, F; h] \text{ hold}\}.
\]

The function \(d_L\) is a metric on \(\Delta^+\). The convergence of the sequence \(\{F_n\}\) of d.d.f.'s to a d.d.f. \(F\) in the metric space \((\Delta^+, d_L)\) is characterized by:

\[
d_L(F_n, F) \to 0 \text{ if and only if } F_n \to F.
\]

Whence, the metric space \((\Delta^+, d_L)\) is compact and complete.

The following statements are direct consequence of definition of \(d_L\).

(i) For any \(F \in \Delta^+\), \(d_L(F, \varepsilon_0) = \inf \{h : F(h+) > 1 - h\}\).
(ii) For any \(t \in (0, 1)\), \(d_L(F, \varepsilon_0) < t\) if and only if \(F(t) > 1 - t\).
(iii) If \(F, G \in \Delta^+\) and \(F \leq G\), then \(d_L(F, \varepsilon_0) \geq d_L(G, \varepsilon_0)\)

A \(t - \text{norm} T\) is a binary operation on \([0, 1]\) that is commutative, associative, non-decreasing in each variable, and has 1 as identity. Among the important example of \(t\)-norms are \(M\) and \(T_L\) respectively defined as:

\[
M(a, b) = \min(a, b) \quad \text{and} \quad T_L(a, b) = \max(a + b - 1, 0).
\]

A triangle function is a mapping \(\tau : \Delta^+ \times \Delta^+ \to \Delta^+\) that is commutative, associative, non-decreasing in each variable, and which has \(\varepsilon_0\) as identity. The typical continuous triangle function is

\[
\tau_T(F, G)(t) = \sup_{u+v=t} T(F(u), G(v)).
\]

The continuity of triangle functions means continuity with respect to the topology of weak convergence in \(\Delta^+\).
Let $X$ be a nonempty set. The mapping $F: X \times X \to \Delta^+$ will be called probabilistic metric and $F(x,y)$ will be denoted by $F_{x,y}$.

**Definition 1.1.** The pair $(X,F)$ is called a Probabilistic Semimetric Space (briefly, a PSM-space) if it is satisfies the following properties:

(P1) $F_{x,x} = \varepsilon_0$; (P2) $F_{x,y} \neq \varepsilon_0$ if $x \neq y$; (P3) $F_{x,y} = F_{y,x}$.

If, in addition the inequality (P4) $F_{x,z} \geq \tau(F_{x,y},F_{y,z})$ takes place, then the triple $(X,F,\tau)$ is called a probabilistic metric space (briefly, a PM-space).

If only the conditions (P1), (P3) and (P4) are fulfilled, then $(X,F,\tau)$ is a probabilistic pseudometric space (briefly, a PPM-space). If $\tau = \tau_T$, then the triple $(X,F,\tau_T)$ is called a Menger space.

Every PSM-space $(X,F)$ endowed with a strong $\lambda$-topology, the topology generated by the strong neighborhood system at $x \in X$, i.e., \{$N_x(\lambda)\}_{\lambda \in (0,1)}$, where $N_x(\lambda) = \{y \in X: d_L(F_{x,y},\varepsilon_0) < \lambda\} = \{y \in X: F_{x,y}(\lambda) > 1 - \lambda\}$.

**Definition 1.2.** Let $(x_n)$ be a sequence in $X$. Then

(i) $x_n \to x$ if and only if $\lim_{n \to \infty} F_{x_n,x} = \varepsilon_0$.

(ii) $(x_n)$ is a Cauchy sequence if and only if $\lim_{m,n \to \infty} F_{x_n,x_m} = \varepsilon_0$.

A PSM-space $(X,F)$ is complete whenever each Cauchy sequence converges to an element of $X$.

**Definition 1.3.** Let $(X,F)$, $(Y,G)$ be PSM-spaces and let $K > 0$. Let $X \to^K Y$ denotes the set of functions $f: X \to Y$ which satisfy the condition:

$$\forall x,y \in X, \forall t \in (0,1) : G_{f(x),f(y)}(t) \geq F_{x,y}(t/K).$$

Note that:

(i) For $K = 1$, the functions are called probabilistic nonexpansive (PNE).

(ii) The elements of $X \to^K Y$, for $t \in (0,1)$ are called probabilistic contractions.

**Lemma 1.4.** Let $Y^X = \{f: X \to Y : f \text{ is PNE}\}$ and $\mathcal{G}_{f,g}(t) = \inf_{x \in X} G_{f(x),g(x)}(t-)$ for every $f,g \in Y^X$. Then, $(Y^X,\mathcal{G},\tau)$ is a PM-space if and only if $(Y,G,\tau)$ is a PM-space.

**Proof:** (P1) For any $f,g \in Y^X$, $\mathcal{G}_{f,g} = \varepsilon_0$ if and only if $\inf_{x \in X} G_{f(x),g(x)} = \varepsilon_0$. Hence, $f = g$, i.e., $f = g$.

(P2) and (P3) are obvious.

Let $f,g,h \in Y^X$. Since $(Y,G,\tau)$ is a PM space, we have $G_{f(x),g(x)} \geq \tau(G_{f(x),h(x)},G_{h(x),g(x)})$. Thus, $\mathcal{G}_{f,g} = \inf_{x \in X} G_{f(x),g(x)} \geq \inf_{x \in X} \tau(G_{f(x),h(x)},G_{h(x),g(x)})$. But, $\inf_{x \in X} \tau(G_{f(x),h(x)},G_{h(x),g(x)}) = \tau(\inf_{x \in X} G_{f(x),h(x)},\inf_{x \in X} G_{h(x),g(x)}) = \tau(\mathcal{G}_{f,h},\mathcal{G}_{h,g})$. Hence, condition (P4) holds.
Definition 1.5. Let \((X, F), (Y, G)\) be PSM-spaces. A function \(f : X \rightarrow Y\) is an isometric embedding if \(G_{fx,fy} = F_{x,y}\) for every \(x, y \in X\). If \(f\) is bijection then it is an isometry.

The following notation will be used: for \(F, F' \in \Delta^+\) and \(\lambda \in (0, 1)\), let

\[ F \approx_\lambda F' \quad \text{if and only if} \quad d_L(F, F') \leq \lambda. \]

(We say that the \(F\) and \(F'\) are equal 'modulo', or 'up to' \(\lambda\))

Definition 1.6. Let \((X, F), (Y, G)\) be PSM-spaces. Two PNE mappings \(f : X \rightarrow Y\) and \(g : Y \rightarrow X\) are \(\lambda\)-adjoint, denoted by \(f \prec_\lambda g\), if for all \(x \in X\) and \(y \in Y\), \(G_{fx,y} \approx_\lambda F_{x,gy}\).

If \(f \prec_0 g\), then \((f, g)\) called a proper adjoint pair, and we shall write \(f \prec g\).

Let \((X, F), (Y, G)\) be PSM-spaces. Consider a pair of PNE mappings \(f : X \rightarrow Y\) and \(g : Y \rightarrow X\), and define \(\varphi_{f,g} = \min\{F_{id_X,gf}, G_{fg,id_Y}\}\).

Intuitively, these d.d.f. gives a probabilistic measure for how similar \((X, F)\) and \((Y, G)\) are.

Definition 1.7. Let \((X, F), (Y, G)\) be PSM-spaces. A pair of PNE mappings \(f : X \rightarrow Y\) and \(g : Y \rightarrow X\) with \(\varphi_{f,g} \approx_\lambda \varepsilon_0\) is called a \(\lambda\)-isometry.

Note that by definition, any pair \((f, g)\) of PNE mappings is an \(\lambda\)-isometry, for \(\varphi_{f,g}(\lambda) > 1 - \lambda\).

The above definition can be justified by the observation that \(+\infty\)-isometries satisfy, due to axiom (P1) in the definition of PM space, \(id_X = g \circ f\) and \(id_Y = f \circ g\), and consequently \(f\) (and also \(g\)) is an isometry.

Let \((X, F, \tau_T), (Y, G, \sigma_T)\) be any PM spaces with \(T \geq T_L\). The following theorem states the equivalence of the notions of \(\lambda\)-adjoint and \(\lambda\)-isometry.

Theorem 1.8. Let \((X, F, \tau_T), (Y, G, \sigma_T)\) be PM spaces, and \(\lambda \in (0, 1)\). For all PNE mappings \(f : X \rightarrow Y\) and \(g : Y \rightarrow X\), \(f \prec_\lambda g\) if and only if \((f, g)\) is an \(\lambda\)-isometry.

Proof: Suppose \(f \prec_\lambda g\). For any \(x \in X\), \(F_{x,gof(x)} \approx_\lambda F_{fx,fx} = \varepsilon_0\), thus \(F_{id_X,gof} \approx_\lambda \varepsilon_0\). Similarly for any \(y \in Y\), \(G_{fog(y),y} \approx_\lambda F_{gy,gy} = \varepsilon_0\), thus \(F_{fog,id_Y} = \varepsilon_0\). It follows that \(\varphi_{f,g} \approx_\lambda \varepsilon_0\). Conversely, suppose \(\varphi_{f,g} \approx_\lambda \varepsilon_0\), then for all
In this section, we present the category of PM spaces whose morphisms are \( \lambda \)-adjoint pairs and introduce the concept of Cauchy tower and initial cone in the probabilistic sense.

**Definition 2.1.** Let \( \text{PMS}^\infty \) denote the category whose objects are non-empty complete PM spaces and \( \lambda \)-adjoint pairs as morphisms. The composition of pair of morphisms \( \iota_1 = (i_1, j_1) \) and \( \iota_2 = (i_2, j_2) \) is defined as \( \iota_1 \circ \iota_2 = (i_2 \circ i_1, j_2 \circ j_1) \).

Notice that if there is a morphism \( \iota = (I, J) : X \to Y \), then we consider \( X \) as an approximation of \( Y \), since \( X \) can be isometrically embedded into \( Y \). The measure of this approximation is given by the d.d.f.

\[
\varphi_{\iota_1 \circ \iota_2} = \inf_{y \in Y} G_{(\iota_1 \circ \iota_2)}(y, y).
\]

**Definition 2.2.** (i) A tower in \( \text{PMS}^\infty \) is a sequence \( (X_n, \iota_n)_n \) of objects and morphisms such that for all \( n \in \mathbb{N} \) we have \( \iota_n : X_n \to X_{n+1} \).

(ii) A tower \( (X_n, \iota_n)_n \) in \( \text{PMS}^\infty \) with \( \iota_n = (i_n, j_n) \), is called a probabilistic Cauchy tower if \( \lim_{m,n \to \infty} \varphi_{\iota_{nm}} = \varepsilon_0 \) where \( \iota_{nm} = \iota_{m-1} \circ \cdots \circ \iota_n \).

The following notation will be used below: For any natural number \( m > n \), \( \iota_{nm} = \iota_{m-1} \circ \cdots \circ \iota_n \) and \( j_{nm} = j_n \circ \cdots \circ j_{m-1} \).

The direct limit of a probabilistic Cauchy tower can be defined as below:
Definition 2.3. The direct limit of \((X_n, \tau_n)_n\) is a cone \((X, (\gamma_n)_n)\), where 
\(\gamma_n = (\alpha_n, \beta_n)\), which is defined as follows:
(a) The space \(X\) is given by \(X = \{(x_n)_n : \forall n \in \mathbb{N}, x_n \in X_n \text{ and } x_n = j_n(x_{n+1})\}\)
is equipped with a probabilistic metric \(F: X \times X \to \Delta^+, \text{ such that for all}\)
\((x_n)_n, (y_n)_n \in X \text{ and } t \in (0, 1)\)
\[F^{X}_{(x_n),(y_n)}(t) = \sup_{n \in \mathbb{N}} F_{x_n,y_n}(t-).\]

(b) Morphisms \(\gamma_n = (\alpha_n, \beta_n): X_n \to X\) are defined as follows:
\(\alpha_n: X_n \to X, \alpha_n(x_n) = (x_k)_k, \text{ where } x_k = \lim_{n \to \infty} (j_{kh} \circ i_{nh}(x_n));\)
\(\beta_n: X \to X_n, \text{ with } \beta_n((x_k)_k) = x_n.\)

Note that \(\alpha_n\) is well defined, because \((j_{kh} \circ i_{nh}(x_n))_{h > \max(k,n)}\) is a Cauchy
sequence. This follows from the fact that \((X_n, \tau_n)_n\) is a Cauchy tower. Further
it is easy to show that \((X, F^X, \tau)\) is a complete PM space and \((X, (\gamma_n)_n)\) is a
cone for the tower \((X_n, \tau_n)_n\).

The notion of initial object of a category can be defined as below:

Definition 2.4. An initial object of a category \(PMS^{\approx}\) is an object in
\(PMS^{\approx}\) such that for every object \(X\) in \(PMS^{\approx}\), there exists a unique mor-
phism \(\iota: A \to X.\)

Proposition 2.5. If \(\lim_{n \to \infty} \varphi_{\gamma_n} = \varepsilon_0,\) then \((X, (\gamma_n)_n)\) will be the initial
cone of the Cauchy tower \((X_n, \tau_n)_n).\)

Proof: Let \((X', (\gamma'_n)_n)\) with \(\gamma'_n = (\alpha'_n, \beta'_n)\) be another cone for \((X_n, \tau_n)_n).\)
We prove the existence of a unique morphism \(\iota: X \to X'\) such that for all \(n \in \mathbb{N}, \gamma'_n = \iota \circ \gamma_n.\) Clearly, \((\alpha'_n \circ \beta_n)_n\) and \((\alpha_n \circ \beta'_n)_n\) are Cauchy sequence because
\((X_n, \tau_n)_n\) is a Cauchy sequence. Since objects of \(PMS^{\approx}\) are complete, we can
define \(i = \lim_{n \to \infty} (\alpha'_n \circ \beta_n)\) and \(j = \lim_{n \to \infty} (\alpha_n \circ \beta'_n).\) This defines a morphism
\(\iota = (i, j): X \to X'.\) It follows from the assumption that \(\lim_{n \to \infty} \varphi_{\gamma_n} = \varepsilon_0\) that
\(\gamma'_n = \iota \circ \gamma_n\) and that \(\iota\) is the unique morphism with this property. This proves
that \((X, (\gamma_n)_n)\) is the initial cone of the tower \((X_n, \tau_n)_n).\)

As a consequence of the above proposition, we have

Corollary 2.6. The direct limit of a Cauchy tower is an initial cone for
that tower.

Proof: This follows from the fact that \(\lim_{n \to \infty} \varphi_{\gamma_n} = \varepsilon_0\) with \((\gamma_n)_n\) as in
Definition 2.3.

The following result gives a criterion for checking the initiality of a cone in
\(PMS^{\approx}.\)
Lemma 2.7. Let \((X_n, \iota_n)_n\) be a Cauchy tower in \(PMS^\approx\) and let \((X, (\gamma_n)_n)\), with \(\gamma_n = (\alpha_n, \beta_n)\), be a cone. Then \((X, (\gamma_n)_n)\) is an initial cone if and only if \(\lim_{n \to \infty} \varphi_{\gamma_n} = \varepsilon_0\).

**Proof:** The proof in the direction of \((\Leftarrow)\) follows from Proposition 2.5. To prove \((\Rightarrow)\), let \((X', (\gamma'_n)_n)\) with \(\gamma'_n = (\alpha'_n, \beta'_n)\) be an initial cone for the Cauchy tower \((X_n, \iota_n)_n\). By the Corollary 2.6, the direct limit \((X, (\gamma_n)_n)\) is initial as well, thus \(X \cong X'\). Therefore \(\lim_{n \to \infty} (\varphi_{\gamma'_n})_n = \lim_{n \to \infty} (\varphi_{\gamma_n})_n = \varepsilon_0\).

3 Fixed point theorem

In this section, we introduce the notion of probabilistic contracting functor which generalizes the notion of probabilistic contracting function. We show that a contracting functor gives rise to a Cauchy tower and that the limit of this tower is a fixed point of a functor.

First, we recall the result of existence and uniqueness of fixed point for B-contracting functions on PM spaces.

**Definition 3.1.** Let \((X, F)\) be a PM space. A function \(f: X \to X\) is called B-contraction if there exists a \(k \in (0, 1)\) such that for every \(x, y \in X\) and \(t > 0\),

\[
F_{fx, fy}(t) \geq F_{x, y}(t^k).
\]

**Theorem 3.2.** (Sehgal-Bharucha’s fixed point theorem) Let \((X, F, \tau)\) be a complete Menger space and \(f: X \to X\) a B-contracting function. Then there exists a unique fixed point \(\text{fix}(f)\) for \(f\) in \(X\) such that

\[
\text{fix}(f) = \lim_{n \to \infty} f^n(x_0), \quad x_0 \in X.
\]

**Definition 3.3.** A functor \(p: PMS^\approx \to PMS^\approx\) is called \(B^\approx\)-contraction if there exists a \(k \in (0, 1)\) such that for each morphism \(\iota: X_1 \to X_2\),

\[
\varphi_{p\iota}(t) \geq \varphi_{\iota}(t^k).
\]

Remark: By the Initiality Lemma, a \(B^\approx\)-contracting functor preserves Cauchy towers and their initial cones, in a similar way as contracting functions preserves Cauchy sequences and their limits.

**Lemma 3.4.** Let \(p: PMS^\approx \to PMS^\approx\) be a \(B^\approx\)-contracting functor and \((X_n, \iota_n)_n\) a Cauchy tower with an initial cone \((X, (\gamma_n)_n)\). Then \((pX_n, p\iota_n)_n\) is a Cauchy tower with \((pX, (p\gamma_n)_n)\) as an initial cone.

The following theorem shows the existence of fixed points for contracting functors on the category \(PMS^\approx\).
Theorem 3.5. Let \( p : PMS^\infty \rightarrow PMS^\infty \) be a \( B^\infty \)-contracting functor. Then \( p \) has a fixed point, i.e., there exists a complete PM space \( X \) such that \( X \cong pX \).

Proof: Let \( X_0 \) be a complete PM space and let \( \iota_0 : X_0 \rightarrow pX_0 \) be any morphism. Consider the tower \( (p^nX_0, p^n\iota_0)_n \). Since \( p \) is \( B^\infty \)-contraction, this is a probabilistic Cauchy tower. Thus it has a direct limit \( (X, (\gamma_n)_n) \), which is an initial cone for the tower. Moreover \( p \) preserves the tower and its initial cone. By the categorical fixed point theorem, \( X \cong pX \).

Notice that contractiveness is not a necessary condition in order that a functor has fixed points as we can see the identity functor is not contracting.

The questions about uniqueness require additional condition of contractiveness on mappings which will be considered in the subsequent paper.

ACKNOWLEDGEMENTS We are gratefully acknowledge the financial support received from the Ministry of Science, Technology and Innovation, Malaysia under Science fund No:06-01-01-SF0177.

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Received: November 5, 2007