Norms of Elementary Operators
in Banach Algebras

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Abstract. Let $T : \Omega \rightarrow \Omega$ be an elementary operator defined by $Tx = \sum_{i=1}^{k} a_i x b_i$ such that $a_i$ and $b_i$ are either in multiplier algebra $M(\Omega)$ or in $\Omega$. We use spectral resolution Theorem to calculate the norm of an elementary operator induced by normal operators in a finite dimensional Hilbert space.

Mathematics Subject Classification: Primary 47B47; Secondary 46A32, 47D20

Keywords: diagonal operators, positive operators, eigenvalues

1. INTRODUCTION

The norm of elementary operators has been a subject of many papers in operator theory. For a $C^*$- algebra $\Omega$, an operator $T : \Omega \rightarrow \Omega$, where $\Omega$ is an algebra, is called an elementary operator if $T$ can be expressed in the form:

$$ Tx = \sum_{i=1}^{k} a_i x b_i $$

with $a_i$ and $b_i$ ($1 \leq i \leq k$) in the multiplier algebra $M(\Omega)$ see [7]. There are various other settings for the definition of elementary operators. For instance,
if $\Omega$ is non-unital then it is more natural to take the coefficients $a_i$ and $b_i$ from the multiplier algebra of $\Omega$ rather than from $\Omega$. An axiomatic approach to the definition of elementary operators has been proposed in [4, 9]. Properties of elementary operators have been investigated in the past two decades under a variety of aspects. There are detailed study of their spectra, see [5], for an overview, such as compactness and norm properties as well as their compatibility with various order relations have been closely examined. Through all these studies, it has emerged that, for general elementary operators, a full description of their properties is rather intricate since these are often intimately interwoven with the structure of the underlying algebras. Therefore there is no general formula known describing the norm of an arbitrary elementary operator, even for simple algebras such as $B(H)$. For a survey on the state-of-the-art of this problem, see [8]. A well known estimate due to Haagerup states that if $Tx = \sum_{i=1}^{k} a_i \otimes b_i$ then

\[
\|T\| \leq \|T\|_{cb} \leq \left\{ \| \sum_{i=1}^{k} a_i a_i^* \| \| \sum_{i=1}^{k} b_i^* b_i \| \right\}^{\frac{1}{2}}
\]

where $\|T\|_{cb}$ is the completely bounded (or $CB$) norm of $T$. we have used simple methods to calculate the norm of elementary operators induced by both diagonal and normal operators by applying spectral resolution Theorem in finite dimensional Hilbert spaces.

2. NORMS OF ELEMENTARY OPERATORS

**Definition 2.1.** Let $\{e_n : n = 0, 1, 2, \ldots n\}$ be an orthonormal basis for $H$. The operator $D$ defined by $De_n = \alpha_n e_n \quad \forall n \in \mathbb{N}$ is called a diagonal operator with diagonals $\{\alpha_n\}$. We say that the operator $D$ is bounded if and only if $|\alpha_n|$ is bounded for all $n \in \mathbb{N}$. In this case we say that this operator is not zero if $\alpha_n \neq 0 \quad \forall n \in \mathbb{N}$.

Let $T : B(H) \rightarrow B(H)$ be an elementary operator. Recall that $T$ is defined as $Tx = \sum_{i=1}^{k} a_i x b_i$. If we let $a_i, b_i \in B(H)$ to be diagonal operators with diagonals $\{\alpha_n\}$ and $\{\beta_n\}$ respectively such that $a_i e_n = \alpha_n e_n \quad \forall n \in \mathbb{N}$ while $b_i e_n = \beta_n e_n \quad \forall n \in \mathbb{N}$. If we further let $H$ to be a finite dimensional space, $\{e_n : n = 1, 2, \ldots\}$ as orthonormal basis of $H$, then we shall assume that $a_i$ and $b_i$ are bounded diagonal operators for $1 \leq i \leq k$. The following Proposition is proved using simple calculation of norm of an elementary operator induced by diagonal operators in a finite complex Hilbert space.

**Theorem 2.2.** Let $T : B(H) \rightarrow B(H)$ be an elementary operator defined by $Tx = \sum_{i=1}^{k} a_i x b_i \quad a_i, b_i \in B(H) \quad x \in B(H)$ where $a_i$ and $b_i$ are diagonal
operators induced by \( \{ \alpha_{ni} \} \) and \( \{ \beta_{ni} \} \) respectively and \( H \) a finite dimensional complex Hilbert space then \( T \) is bounded and

\[
\| T \| = \left( \sum_n \left\{ \sum_{i=1}^k \left| \alpha_{ni} \right|^2 \left| \beta_{ni} \right|^2 \right\} \right)^{\frac{1}{2}}.
\]

Proof. We compute the mapping \( T \) using \( e_n \) as the elements of \( B(H) \). We observe that \( Te_n = \sum_{i=1}^k a_i(e_n)b_i \). Taking \( k \) to be 2 for simplicity we obtain

\[
Te_n = a_1 e_n b_1 + a_2 e_n b_2.
\]

This shows that \( T \) is a diagonal operator with the result that \( Te_n = \sum_n \sum_{i=1}^2 \alpha_{ni} e_n \beta_{ni} \). Generalising therefore we have \( Te_n = \sum_{i=1}^k \alpha_{ni} e_n \beta_{ni} \) which implies that \( T \) is a diagonal operator with diagonals \( \{ \sum_{i=1}^k \alpha_{ni} \beta_{ni} \} \).

From our assumptions in section two, it is clear that this operator is bounded as shown here below

\[
\| Te_n \|^2 = \| \sum_{i=1}^k \alpha_{ni} e_n \beta_{ni} \|^2
\]

\[
\leq \sum_{i=1}^k \sum_{i=1}^k |\alpha_{ni} \beta_{ni}|^2 \| e_n \|^2
\]

\[
= \sum_{i=1}^k \sum_{i=1}^k |\alpha_{ni}|^2 |\beta_{ni}|^2.
\]

But since \( \{ \alpha_{ni} \} \) and \( \{ \beta_{ni} \} \) are bounded \( \forall n \in \mathbb{N} \), it implies that their finite summation is bounded and hence the operator \( T \) is bounded, since taking the suprimum of both sides of the above equation we have

\[
\| T \| \leq \left( \sum_{i=1}^k \left\{ \sum_{i=1}^k \left| \alpha_{ni} \right|^2 \left| \beta_{ni} \right|^2 \right\} \right)^{\frac{1}{2}}.
\]

For the norm of \( T \) consider the calculation

\[
\sum_{n} \sum_{i=1}^k |\alpha_{ni}|^2 |\beta_{ni}|^2 = \| \sum_{n} \sum_{i=1}^k \alpha_{ni} \beta_{ni} \| \| e_n \|^2
\]

\[
= \| \sum_{n} \sum_{i=1}^k \alpha_{ni} e_n \beta_{ni} \| \| e_n \|^2
\]

\[
= \| Te_n \|^2
\]

\[
\leq \| T \|^2 \| e_n \|^2.
\]
Taking the suprimum of both sides since \(\|e_n\| = 1\) we obtain
\[
\sum_n \sum_{i=1}^k |\alpha_{ni}|^2 |\beta_{ni}|^2 \leq \|T\|^2.
\]

On the other hand if this is generalised to an arbitrary \(x \in B(H)\) then we observe that for \(x = \sum_n x_ne_n\) then
\[
Tx = \sum_n x_n T e_n = \sum_n \sum_{i=1}^k x_n \alpha_{ni} e_n \beta_{ni}
\]
so that
\[
\|Tx\|^2 = \sum_n \left| \sum_{i=1}^k x_n \alpha_{ni} e_n \beta_{ni} \right|^2
= \sum_n |x_n|^2 \left| \sum_{i=1}^k \alpha_{ni} e_n \beta_{ni} \right|^2
\leq \sum_n \{ \sum_{i=1}^k |\alpha_{ni}|^2 |\beta_{ni}|^2 \sum_n |x_n|^2 \}
\]
which implies that
\[
\|Tx\|^2 \leq \sum_n \sum_{i=1}^k |\alpha_{ni}|^2 |\beta_{ni}|^2 \|x\|^2.
\]

Taking the suprimum over \(n\) of both sides with \(\|x\| = 1\) we obtain
\[
\|T\|^2 \leq \sum_n \{ \sum_{i=1}^k |\alpha_{ni}|^2 |\beta_{ni}|^2 \}
\]
and with our earlier result we conclude that
\[
\|T\| = \left( \sum_n \{ \sum_{i=1}^k |\alpha_{ni}|^2 |\beta_{ni}|^2 \} \right)^{\frac{1}{2}}.
\]

Since diagonal operators are commutative, we can pass through Haagerup tensor product norms for tensor product \(Tx = \sum_{i=1}^k a_i \otimes b_i\) and \(\|T\|\) is subsequently calculated by application of the following formula
\[
\|T\| = \left\{ \| \sum_{i=1}^k a_i a_i^* \| \| \sum_{i=1}^k b_i^* b_i \| \right\}^{\frac{1}{2}}.
\]
A simple application of the above formula in Theorem 2.2 gives the same result. If \(H = \ell^2\) that is, infinite dimensional complex Hilbert space, then one way to calculate \(\|T\|\) is to note that \(\ell^2\) is unitarily equivalent to the Hilbert
space tensor product $\ell^2 \otimes \ell^2$ from which we can now apply the Haagerup tensor product norm.

**Example 2.3.** As immediate calculation to the above Theorem 2.2, consider the operator $T : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ defined by

$$T(x) = a_1 x b_1 + a_2 x b_2.$$  

Take $x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$, $a_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $a_2 = b_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$Tx = \begin{bmatrix} x_{11} & x_{12} \\ -x_{21} & -x_{22} \end{bmatrix} + \begin{bmatrix} -x_{11} & x_{12} \\ -x_{21} & x_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2x_{12} \\ -2x_{21} & 0 \end{bmatrix}$$

we claim that $\|T\| = 2$. To see this, $(\alpha_{11}, \alpha_{21}) = (1, -1), (\alpha_{12}, \alpha_{22}) = (1, 1), (\beta_{11}, \beta_{21}) = (1, 1), (\beta_{12}, \beta_{22}) = (-1, 1)$. Therefore $\sum_{i=1}^2 |\alpha_{1i}|^2 |\beta_{1i}|^2 = 2$ and $\sum_{i=1}^2 (|\alpha_{2i}|^2 |\beta_{2i}|^2) = 2$ implying that $\|T\| = (\sum_{i=1}^2 |\alpha_{ni}|^2 |\beta_{ni}|^2)^{\frac{1}{2}} = 2$

The following is Proposition for a symmetrised two-sided multiplication operator.

**Proposition 2.4.** Let $T_{a,b} : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$ be a symmetrised two-sided multiplication operator with $a = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$ and $b = \begin{bmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{bmatrix}$ then

$$\|T_{a,b}\| = \sqrt{2\sum_{i=1}^2 |\mu_i|^2 |\nu_i|^2} \cdot \frac{1}{2}, \quad i = 1, 2$$

**Proof.** An operator $T_{a,b}$ is defined by the formula $T_{a,b}x = axb + bxa$ where $x \in M_2(\mathbb{C})$. Therefore if given $x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$, then

$$T_{a,b}(x) = \begin{bmatrix} 2\mu_1 \nu_1 x_{11} + (\mu_1 \nu_2 + \mu_2 \nu_1) x_{12} \\ (\mu_1 \nu_2 + \mu_2 \nu_1) x_{21} + 2\mu_2 \nu_2 x_{22} \end{bmatrix}$$

We can then apply Gram-Schmidt othonormalization to normalize $x$ and if $x$ is normalized then

$$\|T_{a,b}\| = \max\{2|\mu_1||\nu_1|, (|\mu_1 \nu_2 + \mu_2 \nu_1|, 2|\mu_2||\nu_2|)\}.$$  

It follows that when $x$ is diagonal, then $x_{12} = x_{21} = 0$ and hence

$$\|T_{a,b}\| = \max\{2|\mu_1||\nu_1|, 2|\mu_2||\nu_2|\} = (2.1)$$
From Theorem 2.2 using the resultant formula we have

\[ a_1 = a \implies (\alpha_{11}, \alpha_{21}) = (\mu_1, \mu_2) \]
\[ a_2 = b \implies (\alpha_{12}, \alpha_{22}) = (\nu_1, \nu_2) \]
\[ b_1 = b \implies (\beta_{11}, \beta_{21}) = (\nu_1, \nu_2) \]
\[ b_2 = a \implies (\beta_{12}, \beta_{22}) = (\mu_1, \mu_2) \]

Therefore

\[ \sum_{i=1}^{2} |\alpha_{1i}|^2|\beta_{1i}|^2 = |\mu_1|^2|\nu_1|^2 + |\mu_1|^2|\nu_1|^2 = 2|\mu_1|^2|\nu_1|^2 \]

and

\[ \sum_{i=1}^{2} |\alpha_{2i}|^2|\beta_{2i}|^2 = |\mu_2|^2|\nu_2|^2 + |\mu_2|^2|\nu_2|^2 = 2|\mu_2|^2|\nu_2|^2 \]

Hence

\[ \left\| T_{a,b} \right\| = \left( \sum_{i=1}^{2} \sum_{n=1}^{2} |\alpha_{ni}|^2|\beta_{ni}|^2 \right)^{\frac{1}{2}} \]
\[ = (2|\mu_1|^2|\nu_1|^2 + 2|\mu_2|^2|\nu_2|^2)^{\frac{1}{2}} \]
\[ = \sqrt{2}\left( \sum_{i=1}^{2} |\mu_i|^2|\nu_i|^2 \right)^{\frac{1}{2}}, \quad i = 1, 2 \] (2.2)

If \( a \) and \( b \) are normalized such that \( |\mu_i| = |\nu_i| = 1 \quad \forall i = 1, 2 \) then \( |\mu_i\nu_i| = |\mu_i| = |\nu_i| = 1 \) and from equations 2.1 and 2.2 we obtain \( \left\| T_{a,b} \right\| = 2 \). Similarly if \( a \) and \( b \) are such that \( |\mu_i\nu_i| = 1 \quad \forall i = 1, 2 \) then equation 2.1 and 2.2 still results into \( \left\| T_{a,b} \right\| = 2 \).

As immediate calculation to the above Proposition, we consider the following example

**Example 2.5.** Consider \( T_{a,b}x = axb + bxa \) and take

\[ x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}, \quad a = \begin{bmatrix} 2 + i & 0 \\ 0 & 1 - 3i \end{bmatrix}, \quad b = \begin{bmatrix} \frac{1}{5}(2 - i) & 0 \\ 0 & \frac{1}{10}(1 + 3i) \end{bmatrix} \]

Then

\[ \left\| T_{a,b} \right\| = \sqrt{2}\left( \{ |\alpha_{11}|^2|\beta_{11}|^2 + |\alpha_{22}|^2|\beta_{22}|^2 \} \right)^{\frac{1}{2}} = 2. \]

**Remark 2.6.** An application of Proposition 2.4 proves proposition 5.6 in [3] in a simple way and in the case of \( l_2^2 \) with \( a \) and \( b \) being diagonal operators. So here we use separable finite dimensional Hilbert space.
Lemma 2.7 ([11], Lemma 5.6). If \( Z \) is an arbitrary operator in \( H \), then the eigenvalues of \( Z \) constitute a non-empty finite subset of the complex plane. Further more the number of points in this set does not exceed the dimension \( n \) of the space \( H \).

The following Lemma is proved using spectral resolution and more details can be found in [11].

Lemma 2.8. Let \( A \) be a normal operator such that \( A : H \rightarrow H \) where \( H \) is a finite dimensional space then \( \|A\| = (\sum_{j=m} \lambda_j)^{\frac{1}{2}} \) where \( \lambda_j \) are distinct eigenvalues of \( A \) for corresponding eigenspaces \( M_j \), \( j = 1, \ldots, m \).

Proposition 2.9. Let \( T : B(H) \rightarrow B(H) \) be an elementary operator defined by \( Tx = \sum_{i=1}^{k} a_i x b_i \) where \( a_i \) and \( b_i \) are normal operators and \( H \) a finite dimensional Hilbert space then \( \|T\| = (\sum_{i=1}^{k} (\sum_{j=1}^{m} |\alpha_{ij}|^2 |\beta_{ij}|^2))^{\frac{1}{2}} \) where \( \alpha_{ij} \) and \( \beta_{ij} \) are distinct eigenvalues of \( a_i \) and \( b_i \) respectively.

Proof. Since \( T \) is defined from normal operators, \( T \) is normal and by the above Lemma of spectral resolution, for \( a_i \) and \( b_i \) there exists distinct eigenvalues \( \alpha_{ij} \) and \( \beta_{ij} \) respectively and some corresponding eigenspaces for these eigenvalues. By the same Lemma, for every eigenspace, there exist some \( P_{ij} \) and \( Q_{ij} \) pairwise disjoint orthogonal projections for \( a_i \) and \( b_i \) respectively such that

\[
\sum_{i=1}^{k} \sum_{j=1}^{m} P_{ij} = \sum_{i=1}^{k} \sum_{j=1}^{m} Q_{ij} = I
\]

Therefore \( T \) can be expressed as follows

\[
T = \sum_{i=1}^{k} \sum_{j=1}^{m} \alpha_{ij} P_{ij} \beta_{ij} Q_{ij}
\]

and hence

\[
T^* = \sum_{i=1}^{k} \sum_{j=1}^{m} \overline{\alpha_{ij}} P_{ij} \overline{\beta_{ij}} Q_{ij}
\]

Therefore we have

\[
\|TT^*\| = \|T^*T\|
\]

\[
= \sum_{i=1}^{k} \sum_{j=1}^{m} |\alpha_{ij}|^2 |\beta_{ij}|^2 (\sum_{i=1}^{k} \sum_{j=1}^{m} P_{ij})(\sum_{i=1}^{k} \sum_{j=1}^{m} Q_{ij})
\]

\[
= (\sum_{i=1}^{k} \sum_{j=1}^{m} |\alpha_{ij}|^2 |\beta_{ij}|^2)
\]

\[
= \|T\|^2
\]
This implies that
\[ \|T\| = \left( \sum_{i=1}^{k} \sum_{j=1}^{m} |\alpha_{ij}|^2 |\beta_{ij}|^2 \right)^{\frac{1}{2}}. \]

The above Proposition has the following consequence although the proof is from [2], section 6.3.1 but has been included for completion.

**Corollary 2.10.** Let \( T : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \) be an elementary operator defined on normal \( n \times n \) square matrices. Then
\[ \|T\| = \left( \sum_{i=1}^{k} \sum_{j=1}^{n} |p_{ij}|^2 |q_{ij}|^2 \right)^{\frac{1}{2}} \]
where \( p_{ij} \) and \( q_{ij} \) are the elements on the principal diagonals of \( a_i \) and \( b_i \) respectively.

**Proof.** To prove this, we only need to show that for any square matrix \( A \), the sum of the eigenvalues of \( A \) is equal to the sum of the elements on the principal diagonal. If the matrix \( A \) is a null matrix, then there is nothing to be proved. Assume that \( A \) is not null then \( Ax = \lambda x \) and since \( A \) is \( n \times n \) matrix, this equation can be rewritten as \((\lambda I_n - A)x = 0\). This is a system of a homogeneous equation and for non-trivial solution we have that
\[ \det(\lambda I_n - A) = 0 \]
If equation 2.3 is expanded it produces an nth-degree polynomial called a characteristic equation of \( A \) that is
\[ k(\lambda) = \det(\lambda I_n - A) \]
\[ = \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \ldots + k_{n-1} \lambda + k_n = 0 \]
Since the roots of equation 2.4 are \( \lambda_1, \ldots, \lambda_n \) we have
\[ \det(\lambda I_n - A) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) \]
If we set \( \lambda = 0 \) in equation 2.5 we obtain
\[ \det(-A) = (-\lambda_1)(-\lambda_2)\ldots(-\lambda_2) \]
and from the property that \( \det(-A) = (-1)^n \det A \) equation 2.6 reduces to
\[ \det A = \lambda_1 \lambda_2 \ldots \lambda_n = (-1)^n k_n. \]
This shows that the determinant of a matrix is equal to the product of the eigenvalues of that matrix. Next consider the coefficients of \( \lambda^{n-1} \) in equation 2.4. For simplicity take the case \( n=3 \) then
\[ (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) + \ldots \]
and the left hand side of 2.4 becomes

$$det(\lambda I_3 - A) = \begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{bmatrix}$$

Calculating this using first row we have

$$(\lambda - a_{11})A_{11} = (\lambda - a_{11}) \begin{bmatrix} \lambda - a_{22} & -a_{23} \\ -a_{32} & \lambda - a_{33} \end{bmatrix}$$

$$= (\lambda - a_{11})(\lambda^2 - a_{22}\lambda - a_{33}\lambda + ....)$$

$$= \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + ....$$

Hence when \( n = 3 \), comparing the coefficients of \( \lambda^2 \) in equation 2.4 to that of 2.7 gives

$$-(a_{11} + a_{22} + a_{33}) = k = -(\lambda_1 + \lambda_2 + \lambda_3)$$

showing that the sum of the eigenvalues of a matrix is equal to the sum of the elements on the principal diagonal and on application of Proposition 2.9 the required results follows at once.

Since each bounded linear operator can be approximated using matrices, we use the following example to verify our results.

**Example 2.11.** Let \( M_n(\mathbb{C}) \) be the \( C^* \)-algebra of \( n \times n \) square matrices. For \( a_i, b_i \in M_n(\mathbb{C}) \) we show that for \( a_i \) and \( b_i \) to be normal then each is a diagonal matrix and verify that the above formula is valid using some bounded linear matrices.

**Proof.** Without loss of generality we can use \( 2 \times 2 \) square matrices and the proof of \( n \times n \) matrices where \( n > 2 \) can be done in a similar way. Let \( a \) be a \( 2 \times 2 \) matrix ie. \( a = \begin{bmatrix} r & y \\ z & w \end{bmatrix} \). Then we recall that the adjoint of a matrix is the transpose of the the matrix of the cofactors. That is \( a^* = \begin{bmatrix} w & -y \\ -z & r \end{bmatrix} \).

For \( a \) to be normal, it implies that \( aa^* = a^*a \) ie

$$\begin{bmatrix} rw - yz & -ry + yr \\ zw - wz & -zw + wr \end{bmatrix} = \begin{bmatrix} wr - yz & wy - yw \\ -zr + rz & -zy + rw \end{bmatrix}$$

Since the elements of \( \mathbb{C} \) are commutative, we have that \( -ry + yr = zw - wz = -zr + rz = wy - yw = 0 \) and we obtain that \( rw - yz = wr - yz \) implying that \( rw = wr \). If either \( r = 0 \) or \( w = 0 \) or both zero then we obtain a null matrix and the result follows at once since the length of \( T \) is zero and hence \( \|T\| = 0 \).
Assume that $r, w \neq 0$ then $aa^*$ and $a^*a$ are diagonal matrices implying that $a^*, a$ are diagonal. For simplicity let $k = 2$ and

$$a_1 = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}, a_2 = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, b_1 = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}, b_2 = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}.$$ 

If $x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ then

$$Tx = a_1xb_1 + a_2xb_2 = \begin{bmatrix} (\alpha_1\beta_1 + \gamma_1\rho_1)x_{11} & (\alpha_1\beta_2 + \gamma_1\rho_2)x_{12} \\ (\alpha_2\beta_1 + \gamma_2\rho_1)x_{21} & (\alpha_2\beta_2 + \gamma_2\rho_2)x_{22} \end{bmatrix}.$$ 

Therefore assuming that $x$ is diagonal like $a_i$ and $b_i$ we have

$$\sum_{j=1}^{2} |\alpha_{1j}|^2 |\beta_{1j}|^2 = \sum_{m=1}^{2} |\alpha_{mj}|^2 |\beta_{mj}|^2$$

$$\sum_{j=1}^{2} |\alpha_{2j}|^2 |\beta_{2j}|^2 = \sum_{m=1}^{2} |\gamma_{mj}|^2 |\rho_{mj}|^2$$

and therefore

$$\|T\| = \left\{ \sum_{m=1}^{2} (|\alpha_{mj}|^2 |\beta_{mj}|^2 + |\gamma_{mj}|^2 |\rho_{mj}|^2) \right\}^{\frac{1}{2}}$$

which can be compared to results in Theorem 2.2. The Examples 2.3 and 2.5 can be worked out using this formula and the same results can easily be obtained.

**Proposition 2.12.** Let $\Omega$ be a unital prime $C^*$-algebra of positive bounded linear operators which are hermitian or self adjoint. If we define $T : \Omega \rightarrow \Omega$ as an elementary operator then

$$\|T\| = \sum_{i=1}^{k} |\lambda_i|\|a_i\|^2 \quad \forall a_i \in \Omega \quad \lambda_i \in \mathbb{R} \setminus \{0\}$$

with

the property $\|\prod_{i=1}^{k} a_i\| = \prod_{i=1}^{k} \|a_i\|$.

**Proof.** Let $\Omega$ be a prime $C^*$-algebra and $a_i$ and $b_i$ be a collection of linearly independent positive elements of $\Omega$ and $T$ an elementary operator on the algebra $\Omega_h = \{ x \in \Omega : x = x^* \}$. It is clear that $\Omega_h = \Omega$ then $T$ is representable as

$$Tx = \sum_{i=1}^{k} \lambda_i a_i^*xa_i$$

(2.8)
with \( a_i \in \Omega \) linearly independent \( \forall i = 1, 2, ..., k \) and \( \lambda_i (1 \leq i \leq k) \) non-zero real numbers see [7]. Since \( T \) is a hermitian preserving elementary operator the existence of linearly independent operators \( a_i \in \Omega \) that satisfies equation (2.8) is guaranteed by [[7], Corollary 4.9]. Since hermitian operators are self-adjoint, an elementary operator defined from these operators is either self adjoint or not. If the elementary operator is self adjoint then by [[12], Lemma 3.11 and Theorem 3.12],

\[
\|T\| = \|T\|_{cb} = \left\{ \left\| \sum_{i=1}^{k} a_i a_i^* \right\| \left\| \sum_{i=1}^{k} b_i^* b_i \right\| \right\}^{1/2}
\]

and from the above equation we obtain

\[
\|T\| = \sum_{i=1}^{k} |\lambda_i| \left\| a_i a_i^* \right\|
\]

(2.9)

Similarly if \( T \) is not self adjoint then by [6] since \( a_i \) are self adjoint, it implies that \( \|T\| = \|T\|_{cb} \) and in line with the Haagerup tensor norms, equation (2.9) still holds. We also cite [[1], Theorem 5] which also proves that equation (2.9) holds. But \( a_i \in \Omega \) are positive operators and by [[10], Proposition 3.1, Corollary 3.6 & 3.10] and if equality holds in Cauchy-Schwarz type of inequalities that is \( \|\Pi_{i=1}^{k} a_i\| = \Pi_{i=1}^{k} \|a_i\| \) then equation (2.9) reduces to

\[
\|T\| = \sum_{i=1}^{k} |\lambda_i| \left\| a_i a_i^* \right\|
\]

\[
= \sum_{i=1}^{k} |\lambda_i| \left\| a_i \right\|^2
\]

this completes the proof.

Remark 2.13. The above results can also be expressed in terms of \( b_i \in \Omega \) if \( b_i \) is a positive hermitian operator. This is done by scaling \( a_i = \alpha_i b_i^* \) for a suitable scalar \( \alpha_i \in \mathbb{R} \setminus \{0\} \). In that case \( \|T\| = \sum_{i=1}^{k} |\alpha_i|^2 \|b_i\|^2 \). If both \( a_i \) and \( b_i \) are normalized then \( \alpha_i = \lambda_i \quad \forall i = 1, 2, ..., k \) and hence \( \|T\| = \sum_{i=1}^{k} |\lambda_i| \). This confirms the linear dependence between \( a_i \) and \( b_i \).
**Theorem 2.14.** Let $T$ be an elementary operator of minimal length $k \geq 1$ on a unital prime $C^*$-algebra $\Omega$ of positive bounded linear operators. Then $T$ is completely positive if and only if there is a linearly independent subset \( \{c_1, c_2, \ldots, c_k \in \Omega \} \) such that $Tx = \sum_{i=1}^{k} c_i^* x c_i$ and $\|T\| = \sum_{i=1}^{k} \|c_i\|^2$.

**Proof.** The first part of the proof is clear from [[7], Theorem 4.10] and the second part follows by application of Proposition 2.12 ie

$$\|T\| = \| \sum_{i=1}^{k} c_i^* c_i \| = \sum_{i=1}^{k} \|c_i^* c_i\| = \sum_{i=1}^{k} \|c_i\|^2.$$  

\[ \square \]

**Definition 2.15.** A real $n \times n$ matrix $M = [q_{ij}]$ is called non-negative (positive) if all its elements are non-negative (positive) and we write $M \geq 0$.

**Proposition 2.16.** Let $M_n(\mathbb{R})$ be an algebra of positive $n \times n$ diagonal matrices defined as follows $a_i = \text{diag}[\alpha_i]$ and $b_i = \text{diag}[\beta_i]$ then for any elementary operator $T$ defined as $Tx = \sum_{i=1}^{k} a_i x b_i$ where $x \geq 0$ is a positive diagonal matrix which is hermitian and $\beta_i > 0, \alpha_i > 0$ are prime real numbers then $\|T\| = \sum_{i=1}^{k} |\lambda_i| |a_i|^2$; $\lambda_i = \frac{\beta_i}{\alpha_i}$

**Proof.** From the above definition of $a_i$ and $b_i$, it is clear that $a_i$ and $b_i$ are positive elements of $M_n(\mathbb{R})$. It is also clear that $a_i$ and $b_i$ are hermitian matrices and since $x$ is a hermitian positive matrix, $\|T\|$ can be calculated from Proposition 2.12. If $a_i$ and $b_i$ are irreducible, $M_n(\mathbb{R})$ forms a prime $C^*$-algebra. Since $\alpha_i, \beta_i \in \mathbb{R}$, it implies that

$$b_i = \text{diag}[\beta_i] = \text{diag}[\frac{\beta_i}{\alpha_i} \alpha_i] = \frac{\beta_i}{\alpha_i} \text{diag}[\alpha_i] = \frac{\beta_i}{\alpha_i} a_i \quad \frac{\beta_i}{\alpha_i} > 0$$

Therefore

$$\|T\| = \sup_{\|x\|=1} \| \sum_{i=1}^{k} a_i x b_i \|$$

(2.10)

$$= \sup_{\|x\|=1} \| \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} \alpha_i x \alpha_i \|$$

Since $x$ is positive and $x$ is a real diagonal matrix with entries like those of $a_i$ and $b_i$, equation (2.10) above reduces to

$$\|T\| = \sup_{\|x\|=1} \| \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} \alpha_i \| \|x\|$$

(2.11)

$$= \| \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} \alpha_i \|$$
This is because in the definition of $M_n(\mathbb{R})$ in this particular case, equality holds in Cauchy-Schwarz inequality for multiplication. But from the right hand side of equation (2.11), it is clear that $\alpha_i > 0, \frac{\beta_i}{\alpha_i} > 0$ and since they are real numbers, it implies that

$$\|T\| = \left\| \sum_{i=1}^{k} \frac{\beta_i}{\alpha_i} \alpha_i \right\|$$

$$= \sum_{i=1}^{k} \left| \frac{\beta_i}{\alpha_i} \right| \|\alpha_i\alpha_i\|$$

$$= \sum_{i=1}^{k} \left| \frac{\beta_i}{\alpha_i} \right| \|\alpha_i\|^2$$

(2.12)

Taking $\frac{\beta_i}{\alpha_i} = \lambda_i$ and noting that $\|a_i\| = |\alpha_i|$ from the definition of $a_i$, equation (2.12) reduces to

$$\|T\| = \sum_{i=1}^{k} |\lambda_i| \|a_i\|^2$$

which completes the proof.

3. CONCLUSION

The results obtained in this paper indicates that the norm of an elementary operator induced by diagonal operators and normal operators can be calculated using Spectral resolution Theorem in finite-dimensional separable Hilbert spaces. This approach conforms to the results earlier obtained using some other methods and to the best of our knowledge this provides a more direct approach.

REFERENCES


Received: November 20, 2007