A Class of Hyperbolic Equations
with Nonlocal Conditions

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Abstract. In this paper, we study a boundary value nonlocal problem for a class of hyperbolic equations with nonlocal conditions. We prove the uniqueness of the solution, using a priori estimate and also the existence of solution is established by Fourier’s method.

Keywords: Hyperbolic equation, Nonlocal condition, a Priori estimate, Fourier’s method

1. INTRODUCTION

Boundary value problems for parabolic and hyperbolic equations with nonlocal conditions have been investigated in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. Such problems constitute a very interesting and important class. In the present paper, a type of hyperbolic equations with nonlocal conditions is considered. In the rectangular domain \( \Omega = (0, T) \times (0, 1) \), consider the linear hyperbolic equation

\[
\frac{\partial^2 u}{\partial t^2} - \frac{1}{x^r} \frac{\partial}{\partial x} \left( x^{r+2} \frac{\partial u}{\partial x} \right) + ku = 0, \quad k \geq 0 \text{ and } r > 0,
\]
under the initial conditions

\begin{align}
(1.2) & \quad u(0, x) = \phi(x), \\
(1.3) & \quad \frac{\partial u}{\partial t}(0, x) = \psi(x),
\end{align}

and the nonlocal boundary conditions

\begin{align}
(1.4) & \quad \int_0^1 u(t, x)dx = 0, \\
(1.5) & \quad \int_0^1 x^ru(t, x)dx = 0,
\end{align}

where \(\phi(x), \psi(x) \in L_2(0, 1)\) are known functions which satisfy the two compatibility conditions

\[\int_0^1 \phi(x)dx = \int_0^1 x^r\phi(x)dx = 0,\]

and

\[\int_0^1 \psi(x)dx = \int_0^1 x^r\psi(x)dx = 0.\]

The presence of nonlocal conditions raises complications in applying standard methods to solve (1.1)-(1.5). Therefore to overcome this difficulty we will transfer this problem to another which we can handle more effectly. For that, we have the following lemma.

**Lemma 1.** Problem (1.1)-(1.5) is equivalent to the following problem

\[
(Pr)_1 \quad \begin{cases}
\frac{\partial^2 u}{\partial t^2} - \frac{1}{x^r} \frac{\partial}{\partial x} \left( x^{r+2} \frac{\partial u}{\partial x} \right) + ku = 0, \quad k \geq 0 \text{ and } r > 0, \\
\quad \quad \quad u(0, x) = \phi(x), \\
\quad \quad \quad \frac{\partial u}{\partial t}(0, x) = \psi(x), \\
\quad \quad \quad u(t, 1) = 0, \\
\quad \quad \quad \frac{\partial u}{\partial x}(t, 1) = 0.
\end{cases}
\]
Proof. Let \( u(t, x) \) be a solution of (1.1)-(1.5). Integrating Eq.(1.1) with respect to \( x \) over \((0, 1)\), and taking into account of (1.4), we obtain

\[
\left[ x^2 \frac{\partial u}{\partial x} \right]_0^1 + r \int_0^1 x \frac{\partial u}{\partial x} dx = 0,
\]

and so

\[
\frac{\partial u}{\partial x}(t, 1) + ru(t, 1) = 0.
\]

To eliminate the second nonlocal condition \( \int_0^1 x^r u(t, x) dx = 0 \), multiplying both sides of (1.1) by \( x^r \) and integrating the resulting over \((0, 1)\), and taking into account of (1.5), we obtain

\[
\frac{\partial u}{\partial x}(t, 1) = 0.
\]

These may also be written

\[
u(t, 1) = 0,
\]

and

\[
\frac{\partial u}{\partial x}(t, 1) = 0.
\]

Let now \( u(t, x) \) be a solution of \((\text{Pr})_1\), we are required to prove that

\[
\int_0^1 u(t, x) dx = 0,
\]

and

\[
\int_0^1 x^r u(t, x) dx = 0.
\]

We integrate Eq.(1.1) with respect to \( x \), and taking into account of \( u(t, 1) = 0 \) and \( \frac{\partial u}{\partial x}(t, 1) = 0 \), we get

\[
\frac{d^2}{dt^2} \int_0^1 u(t, x) dx + k \int_0^1 u(t, x) dx = 0, \quad t \in (0, T),
\]

and it also follows that

\[
\frac{d^2}{dt^2} \int_0^1 x^r u(t, x) dx + k \int_0^1 x^r u(t, x) dx = 0, \quad t \in (0, T).
\]
By virtue of the compatibility conditions, we get
\[ \int_{0}^{1} u(t, x) dx = 0 \quad \text{and} \quad \int_{0}^{1} x^r u(t, x) dx = 0. \]

2. A PRIORI ESTIMATE

Consider the function space
\[ E = \left\{ u : u \in L_2(0, 1), \frac{\partial u}{\partial t} \in L_2(0, 1), x \frac{\partial u}{\partial x} \in L_2(0, 1) \right\}, \]
where \( \Omega^r = (0, \tau) \times (0, 1) \), with respect to the norm
\[ \| v \|_E^2 = \sup_{\tau \in (0, T)} \int_{0}^{1} \left[ u^2(\tau, x) + \left( \frac{\partial u}{\partial t} \right)^2(\tau, x) + x^2 (\frac{\partial u}{\partial x})^2(\tau, x) \right] dx. \]

Note that \( E \) is a Hilbert space.

We will establish an energy inequality which ensures the uniqueness of the solution of (Pr)\(_1\) in \( E \).

**Theorem 1.** For problem (1.1)-(1.5) (Pr)\(_1\), we have
\[ \| u \|_E \leq c \left( \| \phi \|_{H^1(0, 1)} + \| \psi \|_{L_2(0, 1)} \right), \]
where \( c > 0 \) is independent of \( u \).

**Proof.** Consider the scalar product \( (\ell u, 2\frac{\partial u}{\partial t})_{L_2(\Omega^r)} \), where \( \ell u = 0 \) and
\[ \ell u \equiv \frac{\partial^2 u}{\partial t^2} - \frac{1}{x^r \frac{\partial}{\partial x}} \left( x^{r+2} \frac{\partial u}{\partial x} \right) + ku. \]

Employing integration by parts, and taking into account initial and boundary conditions of (Pr)\(_1\), we obtain
\[ \int_{0}^{1} \left[ ku^2 + \left( \frac{\partial u}{\partial t} \right)^2 + x^2 \left( \frac{\partial u}{\partial x} \right)^2 \right] (\tau, x) dx = 2r \int_{\Omega^r} x \frac{\partial u}{\partial x} \frac{\partial t}{\partial t} dt dx + k \int_{0}^{1} \phi^2(x) dx + \int_{0}^{1} \psi^2(x) dx + \int_{0}^{1} x^2 \phi^2(x) dx. \]
We now apply the $\varepsilon$- inequality $2 \left| ab \right| \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$, $\varepsilon > 0$ to the first term of the RHS of last equality to obtain
\[
\int_0^1 \left[ k u^2 + \left( \frac{\partial u}{\partial t} \right)^2 + x^2 \left( \frac{\partial u}{\partial x} \right)^2 \right] (\tau, x) \, dx \leq \frac{r}{\varepsilon_1} \int_{\Omega^\tau} x^2 \left( \frac{\partial u}{\partial x} \right)^2 (t, x) \, dx + r \varepsilon_1 \int_{\Omega^\tau} \left( \frac{\partial u}{\partial t} \right)^2 (t, x) \, dx + k \int_0^1 f^2 (x) \, dx + \int_0^1 \psi^2 (x) \, dx + \int_0^1 x^2 f' (x) \, dx.
\]

Since $x \leq 1$ and $\Omega^\tau \subset \Omega$, and taking $k_1 = 1 - r \varepsilon_1 > 0$ and $k_2 = 1 - r \frac{1}{\varepsilon_1} > 0$, we have
\[
\int_0^1 \left[ k u^2 + k_1 \left( \frac{\partial u}{\partial t} \right)^2 + k_2 x^2 \left( \frac{\partial u}{\partial x} \right)^2 \right] (\tau, x) \, dx \leq k \int_0^1 f^2 (x) \, dx + \int_0^1 \psi^2 (x) \, dx + \int_0^1 x^2 f' (x) \, dx.
\]

Now, the right-hand side of this last inequality is independent of $\tau$, and replacing the left-hand side by its upper bound with respect to $\tau$ in the interval $(0, T)$, we obtain the desired inequality which $c = \frac{\max(1, k)}{\min(k, k_1, k_2)}$.

This completes the proof. $\square$

3. EXISTENCE OF SOLUTION

To find the solution of (1.1)-(1.5) we make use of the Fourier's method. Let $u_n (t, x) = T_n (t) X_n (x)$, where $X_n (x)$ is an eigenfunction of the following problem

\[
\begin{cases}
\frac{1}{x^r} \frac{d}{dx} \left( x^{r+2} \frac{dX_n}{dx} \right) + \left( \lambda_n^2 - k \right) X_n = 0, \\
X_n (1) = 0, \\
\frac{dX_n}{dx} (1) = 0,
\end{cases}
\]

where $\lambda_n$, $n = 1, 2, ...$ is called the eigenvalue corresponding to the eigenfunction $X_n (x)$, while $T_n (t)$ is a function satisfying the following equation

\[(3.1)\quad \frac{d^2 T_n}{dt^2} + \lambda_n^2 T_n = 0.\]

So we have

\[T_n (t) = a_n \cos \lambda_n t + b_n \sin \lambda_n t.\]

Regarding the eigenfunction $X_n (x)$ we get

**Lemma 2.**

\[(3.2)\quad \int_0^1 x^r X_n (x) X_m (x) \, dx = 0, \quad n \neq m.\]
Proof. Let $X_n$ and $X_m$ be two eigenfunctions corresponding to the eigenvalues $\lambda_n$ and $\lambda_m$ respectively and satisfy

\begin{equation}
\frac{1}{x^r} \frac{d}{dx} \left( x^{r+2} \frac{dX_n}{dx} \right) + \left( \lambda_n^2 - k \right) X_n = 0,
\tag{3.3}
\end{equation}

with

\begin{equation}
X_n(1) = 0, \quad \frac{dX_n}{dx}(1) = 0,
\tag{3.4}
\end{equation}

and

\begin{equation}
\frac{1}{x^r} \frac{d}{dx} \left( x^{r+2} \frac{dX_m}{dx} \right) + \left( \lambda_m^2 - k \right) X_m = 0,
\tag{3.5}
\end{equation}

with

\begin{equation}
X_m(1) = 0, \quad \frac{dX_m}{dx}(1) = 0.
\tag{3.6}
\end{equation}

Multiplying both sides of (3.3) and (3.5) by $X_m$ and $X_n$ respectively, then summing side to side the results, we get

\begin{equation}
\frac{d}{dx} \left( x^{r+2} \frac{dX_n}{dx} \right) X_m - \frac{d}{dx} \left( x^{r+2} \frac{dX_m}{dx} \right) X_n = (\lambda_m^2 - \lambda_n^2) x^r X_n X_m.
\end{equation}

Then integrating by parts from 0 to 1, we obtain

\begin{equation}
(\lambda_m^2 - \lambda_n^2) \int_0^1 x^r X_n(x) X_m(x) dx = \frac{dX_n}{dx}(1) X_m(1) - \frac{dX_m}{dx}(1) X_n(1).
\end{equation}

Hence using (3.4) and (3.6), we get

\begin{equation}
\int_0^1 x^r X_n(x) X_m(x) dx = 0, \quad \lambda_m^2 \neq \lambda_n^2.
\end{equation}

This completes the proof.

By principal of superposition, the solution of (1.1) – (1.5) is given by the series

\begin{equation}
u(t, x) = \sum_{n=1}^{\infty} X_n(x) (a_n \cos \lambda_n t + b_n \sin \lambda_n t).
\tag{3.7}
\end{equation}

We now return to initial conditions (1.2) – (1.3), to get

\begin{equation}
u(0, x) = \sum_{n=1}^{\infty} a_n X_n(x) = \phi(x), \quad 0 < x < 1,
\end{equation}

\end{document}
\[
\frac{\partial u}{\partial t}(0, x) = \sum_{1}^{\infty} b_n \lambda_n X_n(x) = \psi(x), \quad 0 < x < 1.
\]

To find the constants \(a_n\) and \(b_n\) we use Lemma 3 to get
\[
a_n = \frac{\int_{0}^{1} \phi(x)x^r X_n(x)dx}{\int_{0}^{1} x^r X_n^2(x)dx}
\]
and
\[
b_n = \frac{\int_{0}^{1} \psi(x)x^r X_n(x)dx}{\lambda_n \int_{0}^{1} x^r X_n^2(x)dx}.
\]

Thus we have

**Theorem 2.** Let \(\phi \in H^1(0,1)\) and \(\psi \in L_2(0,1)\). Then there exists a solution of (1.1)-(1.5) which has the form of a sum of (3.7).

**References**


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