Compact Composition Operator on Weighted Bergman-Orlicz Space

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Abstract

In this paper we study the weighted Bergman-Orlicz spaces $A^{\varphi}_\alpha$. Among other properties we get that $A^{\varphi}_\alpha$ is a Banach space with the Luxemburg norm. We show that the set of analytic polynomials is dense in $A^{\varphi}_\alpha$. We also study compactness and continuity of the composition operator on $A^{\varphi}_\alpha$.

Mathematics Subject Classification: 46E30, 47B33

Keywords: Bergman-Orlicz spaces, Composition operator

1 Introduction

Let $\varphi : [0, \infty) \to [0, \infty)$ be a increasing and convex function satisfying $\varphi(0) = 0$, we will assume that $\varphi$ satisfies a $\Delta_2$ condition globally, that is, for every $r \geq 0$, there exists a positive constant $K_1$, depending only on $r$, such that

$$\varphi(rt) \leq K_1(r)\varphi(t), \ \forall t \in [0, \infty).$$

(1)

For instance, $\varphi_1(t) = t^p$ and $\varphi_2(t) = t^p \log(2 + t)$ with $p \geq 1$ satisfy (1); in the second case we can choose $K_1(r) = r^p$, for $r \leq 1$ and $K_1(r) = r^{p+1}$ for $r > 1$.

Given a function $\varphi$ which satisfies the properties mention above and $\alpha > -1$, we can define the weighted Bergman-Orlicz spaces $A^{\varphi}_\alpha$ as the set of all analytic
functions $f$ on the unit disk $\mathbb{D}$ such that

$$M_{\alpha,\varphi}(f) := \int_{\mathbb{D}} \varphi(|f(z)|) \, dA_\alpha(z) < \infty,$$

(2)

where $dA_\alpha(z) = (1 + \alpha)(1 - |z|^2)^\alpha \, dA(z)$ and $dA(z) = \frac{1}{\pi} dx dy = r \, dr \, d\theta$ is the normalized bidimensional Lebesgue's measure with $z = x + iy = re^{i\theta}$.

Using (1) and the fact that $\varphi$ is an increasing convex function, we can see that, if $\lambda \in \mathbb{C}$ and $f, g \in A^{\varphi}_\alpha$, then

$$M_{\alpha,\varphi}(\lambda f + g) \leq \int_{\mathbb{D}} \varphi(|\lambda| |f(z)| + |g(z)|) \, dA_\alpha(z)$$

$$\leq K_2 (|\lambda|) M_{\alpha,\varphi}(f) + M_{\alpha,\varphi}(g),$$

where $K_2$ is a constant. Thus, from this last inequality, we conclude that $A^{\varphi}_\alpha$ is a vector space on $\mathbb{C}$. Moreover, if $\varphi(t) = t^p$ with $t \geq 0$ and $p \geq 1$, we get back the classical weighted Bergman space $A^p_\alpha$ (see [2], [4] and [10] for references of Bergman spaces). It is not hard to prove that $A^{\varphi}_\alpha$ equipped with Luxemburg's norm

$$\|f\|_{\alpha,\varphi} := \inf \left\{ k > 0 : M_{\alpha,\varphi}\left(\frac{f}{k}\right) \leq 1 \right\},$$

is a normed space.

Different type of Bergman-Orlicz spaces has been studied by other authors, for instance in [9] the authors consider a $N$-function, that is, a convex function which satisfies $\varphi(0) = 0$, $\lim_{t \to 0^+} \varphi(t) / t = 0$ and $\lim_{t \to \infty} \varphi(t) / t = \infty$. They defined and studied some properties of the unweighted Bergman-Orlicz spaces

$$L^{\varphi^*}_\alpha = \{ u \in L^{\varphi^*} : u \in H(\mathbb{D}) \},$$

where $L^{\varphi^*}$ denote the Orlicz spaces. Similarly, in [6, 7], the authors consider a modulus function $\psi : [0, \infty) \to [0, \infty)$, that is, a strictly increasing, subadditive and right continuous function such that $\psi(t) = 0$ if and only if $t = 0$. Also, they defined

$$A^\psi = \left\{ f \in H(\mathbb{D}) : \|f\|_\psi = \int_{\mathbb{D}} \psi(|f|) \, dA(z) < \infty \right\},$$

(Stevic added a weight see [7]). Under certain conditions on the function $\psi$ they obtained a complete metric space, moreover they studied some properties of the composition operator.

Recently, numerous authors (see [3], [5] and the references cited there) have defined the Bergman-Orlicz spaces, $A_{\varphi}(\nu_\alpha)$, on unit ball $B$ of $\mathbb{C}^n$ as the set of all analytic functions $f$ on $B$ such that

$$\int_B \varphi(\log |f(z)|) \, d\nu_\alpha(z) < \infty.$$
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where $\phi : \mathbb{R} \to [0, \infty)$ is a $C^2$ nondecreasing convex function, $d\nu_\alpha(z) = \left(1 - |z|^2\right)^\alpha d\nu(z)$ and $d\nu$ denotes the normalized Lebesgue measure on the ball $B$. Employing the Riesz measure, the authors in [3] and [5] get characterizations of these spaces similar to the characterizations given by Stoll [8] for Hardy-Orlicz spaces.

In this paper we study the composition operator on weighted Bergman-Orlicz spaces $A^\varphi_\alpha$. In the section 2, we will modify the technique used in [10] to obtain some important properties about the growth of functions in $A^\varphi_\alpha$ (see Theorem 2.1). This property will allow us to establish that $A^\varphi_\alpha$ is a Banach spaces with the Luxemburg norm (see Theorem 2.2). In the section 3, we will study compactness and continuity of the composition operator on $A^\varphi_\alpha$.

2 Properties of $A^\varphi_\alpha$

In this section we briefly state and prove a few basic results regarding weighted Bergman-Orlicz space $A^\varphi_\alpha$.

**Theorem 2.1** Let $K$ be a compact subset of $\mathbb{D}$ and $n \in \mathbb{N} \cup \{0\}$. Then there exists a constant $C = C(\alpha, K, n) > 0$ such that

$$
\sup_{z \in K} \{ \varphi \left( |f^{(n)}(z)| \right) \} \leq CM_{\alpha, \varphi}(f),
$$

for all $f \in A^\varphi_\alpha$.

**Proof.** Let $f \in A^\varphi_\alpha$, we consider the cases $n = 0$ and $n \geq 1$ separately.

Case I: $n = 0$. Let $z \in \mathbb{D}$ and $f \in A^\varphi_\alpha$, since $|f|$ is subharmonic function and $\varphi$ is an increasing convex function, we can see that $\varphi(|f|)$ is a subharmonic function. Thus

$$
\varphi \left( |f(z)| \right) \leq \frac{1}{r^2} \int_{B(z, r)} \varphi \left( |f(w)| \right) dA(w),
$$

where $r = 1 - |z|$; but if $w \in B(z, r)$, then considering the cases $\alpha \geq 0$ and $\alpha < 0$, by virtue of the triangle inequality we obtain a constant $C(\alpha) > 0$ such that

$$
(1 - |w|^2)^\alpha \geq C(\alpha) \left(1 - |z|^2\right)^\alpha.
$$

Now, employing the latter inequality in (4) we have

$$
\varphi \left( |f(z)| \right) \leq \frac{1}{C(\alpha)r^{2+\alpha}} \int_{B(z, r)} \varphi \left( |f(w)| \right) dA_\alpha(w),
$$

$$
\leq \frac{1}{C(\alpha) \left(1 - |z|\right)^{2+\alpha} M_{\alpha, \varphi}(f)}.
$$

(5)
Next, since the function \((1 - |z|)^{2+\alpha}\) is continuous for all \(z \in K\) we can find a constant \(C = C(\alpha, K)\) such that (3) holds.

Case II: \(n \geq 1\). Let \(z \in \mathbb{D}\) and \(R = \frac{1}{2}(1 - |z|)\), then the function \(f\) is analytic on the closed disk \(\overline{B}(z, R)\). Now, if \(z \in K\), then by the Cauchy integral formula

\[
 f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\lambda - z| = R} \frac{f(\lambda)}{(\lambda - z)^{n+1}} d\lambda.
\]

We can find a constant \(C_K > 0\), such that

\[
 |f^{(n)}(z)| \leq C_K \int_0^{2\pi} |f(z + Re^{i\theta})| \frac{d\theta}{2\pi}.
\]

Next, applying the Jensen inequality, together with the fact that \(\varphi\) is an increasing convex function satisfying (1), we obtain

\[
 \varphi \left( |f^{(n)}(z)| \right) \leq K_1(C_K) \varphi \left( \int_0^{2\pi} |f(z + Re^{i\theta})| \frac{d\theta}{2\pi} \right)
\]

\[
 \leq \frac{1}{2\pi} K_1(C_K) \int_0^{2\pi} \varphi \left( |f(z + Re^{i\theta})| \right) d\theta
\]

\[
 \leq CM_{\alpha, \varphi}(f),
\]

where we have used the case I in the last inequality. This complete the proof of Theorem 2.1.

**Theorem 2.2** \((A_{\alpha}^{\varphi}, \| \cdot \|_{\alpha, \varphi})\) is a Banach space.

**Proof.** We consider an arbitrary Cauchy sequence \(\{f_n\}\) in \(A_{\alpha}^{\varphi}\). Next, we will show that \(\{f_n\}\) converges to a function \(f \in A_{\alpha}^{\varphi}\). Let \(K\) be a compact subset of \(\mathbb{D}\), \(z \in K\), \(\varepsilon > 0\) and \(C > 0\) the constant which is found in the Theorem 2.1. We choose \(\varepsilon > 0\) such that \(\varepsilon \varphi^{-1}(C) < \varepsilon\), since \(\{f_n\}\) is Cauchy, there is an \(n_0 \in \mathbb{N}\) such that \(\|f_n - f_m\|_{\alpha, \varphi} < \varepsilon\) whenever \(n, m \geq n_0\). By the definition of the Luxemburg’s norm, there exists \(k_0 > 0\) such that \(k_0 < \varepsilon\) and

\[
 M_{\alpha, \varphi} \left( \frac{f_n - f_m}{k_0} \right) \leq 1. \quad (6)
\]

Now, Theorem 2.1, our selection of \(C\), Theorem 2.1 and the relation (6) implies

\[
 \varphi \left( \frac{1}{k_0} |f_n(z) - f_m(z)| \right) \leq C;
\]

for \(z \in K\). Thus

\[
 |f_n(z) - f_m(z)| \leq k_0 \varphi^{-1}(C) < \varepsilon \varphi^{-1}(C) < \varepsilon,
\]
for all \( z \in K \) whenever \( n, m \geq n_0 \).

This shows that \( \{f_n\} \) is an uniformly Cauchy sequence on compact subsets of \( \mathbb{D} \), thus there exists a function \( f \in H(\mathbb{D}) \) such that \( \{f_n\} \) converge uniformly to \( f \) on compact subsets of \( \mathbb{D} \). We now invoke the Fatou’s lemma and (6) to obtain

\[
M_{\alpha, \varphi}\left( \frac{f - f_m}{k_0} \right) = \int_{\mathbb{D}} \liminf_{n \to \infty} \varphi\left( \frac{|f_n(z) - f_m(z)|}{k_0} \right) dA_\alpha(z) \\
\leq \liminf_{n \to \infty} \int_{\mathbb{D}} \varphi\left( \frac{|f_n(z) - f_m(z)|}{k_0} \right) dA_\alpha(z) \leq 1.
\]

Thus \( \frac{1}{k_0} (f - f_m) \in A_{\alpha}^{\varphi} \), and consequently \( f \in A_{\alpha}^{\varphi} \). This proves completeness of \( A_{\alpha}^{\varphi} \).

**Theorem 2.3** The set of analytic polynomials is dense in \( A_{\alpha}^{\varphi} \).

**Proof.** Let \( f \in A_{\alpha}^{\varphi} \) and \( \varepsilon > 0 \). For \( 0 < \rho < 1 \), we set \( f_\rho(z) = f(\rho z) \) with \( z \in \mathbb{D} \). We next claim

\[
\lim_{\rho \to 1^{-}} \|f - f_\rho\|_{\alpha; \varphi} = 0.
\]

(7)

To prove this, we choose \( \bar{\varepsilon} > 0 \) such that

\[
K_1\left( \frac{1}{\varepsilon} \right) \{1 + K_1(2)\} \bar{\varepsilon} < 1.
\]

Since \( f \in A_{\alpha}^{\varphi} \), there exists \( R \in (0, 1) \) such that

\[
\int_{A_R} \varphi(|f|) dA_\alpha < \bar{\varepsilon},
\]

(8)

where \( A_R = \{z \in \mathbb{D} : |z| > R\} \).

Next, in view of the fact that the mean of subharmonic functions increases with the radius, we can see that (8) holds if we replace \( f \) by \( f_\rho \). Thus, using (1) and the fact that \( \varphi \) is a convex and increasing function we obtain

\[
\int_{A_R} \varphi\left( \frac{1}{\varepsilon} |f - f_\rho| \right) dA_\alpha \leq K_1\left( \frac{1}{\varepsilon} \right) \int_{A_R} \varphi(|f - f_\rho|) dA_\alpha \\
\leq K_1\left( \frac{1}{\varepsilon} \right) K_1(2) \int_{A_R} \varphi\left( \frac{1}{2} |f| + \frac{1}{2} |f_\rho| \right) dA_\alpha \\
\leq K_1\left( \frac{1}{\varepsilon} \right) K_1(2) \left\{ \int_{A_R} \varphi(|f|) dA_\alpha + \int_{A_R} \varphi(|f_\rho|) dA_\alpha \right\} \\
< K_1\left( \frac{1}{\varepsilon} \right) K_1(2) \bar{\varepsilon}.
\]

(9)
On the other hand, since \( \lim_{t \to 0^+} \varphi(t) = 0 \), we can find \( \delta > 0 \) such that \( \varphi(\delta) < \varepsilon \). Notice that \( f \) is uniformly continuous on \( D_R = \{ z \in \mathbb{D} : |z| \leq R \} \); so, we can find a \( \rho_0 < 1 \) such that

\[
|f(z) - f_\rho(z)| < \delta,
\]

for all \( z \in D_R \) and all \( \rho_0 < \rho < 1 \). Therefore, for \( \rho \in (\rho_0, 1) \) we have

\[
\int_{D_R} \varphi \left( \frac{1}{\varepsilon} |f - f_\rho| \right) dA_\alpha \leq K_1 \left( \frac{1}{\varepsilon} \right) \int_{D_R} \varphi (|f - f_\rho|) dA_\alpha
\]

\[
\leq K_1 \left( \frac{1}{\varepsilon} \right) \varphi (\delta)
\]

\[
< K_1 \left( \frac{1}{\varepsilon} \right) \varepsilon.
\]

By (9) and (10) we obtain \( \|f - f_\rho\|_{\alpha, \varphi} < \varepsilon \) whenever \( \rho_0 < \rho < 1 \). This complete the proof of our claim.

Finally, for \( \rho \in (\rho_0, 1) \) fixed, the function \( f_\rho \) is analytic on \( \mathbb{D} \), then its Taylor’s series converge uniformly to \( f_\rho \) over the disk \( \mathbb{D} \), thus, choosing \( \delta > 0 \) such that \( \varphi(\delta) < 1 \), there exists a polynomial \( P \) such that \( |f_\rho(z) - P(z)| < \varepsilon \delta \) for all \( z \in \mathbb{D} \) and so we have

\[
M_{\alpha, \varphi} \left( \frac{1}{\varepsilon} (f_\rho - P) \right) \leq \varphi(\delta) < 1.
\]

This means that \( \|f_\rho - P\|_{\alpha, \varphi} < \varepsilon \) and the proof is finished.

## 3 Composition operator on \( A^\varphi_\alpha \)

For each \( f \in A^\varphi_\alpha \), the composition operator is defined by \( C_\varphi (f) = f \circ \phi \), where \( \phi \) is an analytic function of \( \mathbb{D} \) into itself. In this section we will prove that \( C_\phi \) is a continuous linear operator from \( A^\varphi_\alpha \) into itself. Also we study its compactness.

We now are ready to state and prove the following

**Theorem 3.1** Suppose that \( \phi : \mathbb{D} \to \mathbb{D} \) is analytic, then

\[
\int_{\mathbb{D}} \varphi (|f \circ \phi|) dA_\alpha \leq \left( \frac{1 + |\phi(0)|}{1 - |\phi(0)|} \right)^{\alpha+2} \int_{\mathbb{D}} \varphi (|f|) dA_\alpha,
\]

for all \( f \in A^\varphi_\alpha \).

**Proof.** On the one hand, let us define \( a = \phi(0) \), \( \phi_a(z) = \frac{a - z}{1 - \bar{a} z} \) with \( z \in \mathbb{D} \) and the function \( g \) given by \( g(z) = \phi_a \circ \phi \). It is not hard to see that \( g \)
is an analytic function on $\mathbb{D}$ into itself such that $g(0) = 0$. On the other hand, since $\varphi (|f \circ \phi_a|)$ is a continuous function on the circle $|z| = r$ with $r \in (0, 1)$, there exists a harmonic function $h$ on $D_r = \{z \in \mathbb{D} : |z| < r\}$ such that $h(z) = \varphi (|f \circ \phi_a(z)|)$ for all $z \in \partial D_r$, moreover, since $|f \circ \phi_a|$ is a sub-harmonic function on $D_r$ and $\varphi$ is increasing and convex function, we obtain that $\varphi (|f \circ \phi_a|)$ is a subharmonic function on $D_r$. By the maximum principle we have

$$\varphi (|f \circ \phi_a(z)|) \leq h(z), \quad \forall z \in \overline{D_r}. \quad (12)$$

Schwarz’s lemma allows us to replace $z$ by $g(z)$ in (12) because the function $g$ maps the disk $D_r$ into itself, thus we have

$$\varphi (|f \circ \phi_a \circ g(z)|) \leq h (g(z)), \quad \forall z \in \overline{D_r}. \quad (13)$$

Since $g$ is analytic and $h$ is harmonic it is easy to see that the function $h \circ g$ is harmonic on $D_r$, hence, by (13) and the mean value property, we can write

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi (|f \circ \phi (re^{i\theta})|) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \varphi (|f \circ \phi_a \circ g (re^{i\theta})|) \, d\theta$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} h \circ g (re^{i\theta}) \, d\theta$$

$$= h (g(0)) = h(0)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} h (re^{i\theta}) \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \varphi (|f \circ \phi_a (re^{i\theta})|) \, d\theta,$$

where we have used the fact that $\phi = \phi_a \circ g$ since $\phi_a^{-1} = \phi_a$. Next, multiplying both sides of the above inequality by $r (1 - r^2)^\alpha$ and integrating with respect to $r$ we obtain

$$\int_{\mathbb{D}} \varphi (|f \circ \phi|) \, dA_a \leq \int_{\mathbb{D}} \varphi (|f \circ \phi_a|) \, dA_a. \quad (14)$$

Making the change of variable $w = \phi_a(z)$ whose Jacobian satisfy $dA(z) = |\phi_a'(w)|^2 \, dA(w)$ and using the fact that $1 - |\phi_a(w)|^2 = (1 - |w|^2) |\phi_a'(w)|$ we have

$$\int_{\mathbb{D}} \varphi (|f \circ \phi(z)|) \, dA_a(z) \leq \int_{\mathbb{D}} \varphi (|f(w)|) (1 - |\phi_a(w)|^2)^\alpha |\phi_a'(w)|^2 \, dA(w)$$

$$= \int_{\mathbb{D}} \varphi (|f(w)|) |\phi_a'(w)|^{\alpha+2} \, dA_a(w)$$

$$\leq \left(\frac{1 + |a|}{1 - |a|}\right)^{2+\alpha} \int_{\mathbb{D}} \varphi (|f(w)|) \, dA_a(w).$$
This complete the proof of Theorem 3.1.

Now, we give necessary and sufficient conditions in order that \( C_\phi \) will be a compact operator. Let us recall the following well-known result. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two complete metric spaces and \( T : \mathcal{X} \to \mathcal{Y} \) a operator. \( T \) is compact if and only if it maps every bounded sequence \( \{x_n\} \) in \( \mathcal{X} \) onto a sequence \( \{T(x_n)\} \) in \( \mathcal{Y} \) which has a convergent subsequence.

**Theorem 3.2** \( C_\phi \) is compact on \( A_\alpha^\varphi \) if and only if for each bounded sequence \( \{f_n\} \) in \( A_\alpha^\varphi \) such that \( f_n \to 0 \) uniformly on compact subsets of \( \mathbb{D} \) holds, \( \| f_n \circ \phi \|_{\alpha, \varphi} \to 0 \) as \( n \to \infty \).

**Proof.** Suppose that \( C_\phi \) is a compact operator. Let \( \{f_n\} \subset A_\alpha^\varphi \) be a bounded sequence of elements in \( A_\alpha^\varphi \) such that \( f_n \to 0 \) uniformly on every compact subset of \( \mathbb{D} \). Then, by compactness of \( C_\phi \), the sequence \( \{C_\phi(f_n)\} \) has a subsequence which converges to \( f \in A_\alpha^\varphi \). Moreover, for each \( z \in \mathbb{D} \) the unitary set \( K = \{\phi(z)\} \) is compact, then we must have \( f_n \circ \phi(z) \to 0 \) and so \( f \equiv 0 \). Therefore, by the continuity of the norm we conclude that \( \lim_{n \to \infty} \| f_n \circ \phi \|_{\alpha, \varphi} = 0 \).

For the converse, suppose that \( \{f_n\} \) is a bounded sequence in \( A_\alpha^\varphi \), then there exists \( M > 0 \) such that \( \| f_n \|_{\alpha, \varphi} \leq M \) for all \( n \in \mathbb{N} \). Our next goal is to show that \( \{C_\phi(f_n)\} \) has a convergent subsequence in \( A_\alpha^\varphi \). To this end, let \( K \) be a compact subset of \( \mathbb{D} \) and \( z \in K \). By definition of Luxemburg’s norm, the condition \( \| f_n \|_{\alpha, \varphi} \leq M \) for all \( n \in \mathbb{N} \) implies that

\[
M_{\alpha, \varphi} \left( \frac{f_n}{M} \right) \leq 1, \quad \forall n \in \mathbb{N};
\]

then by Theorem 2.1, there exists a constant \( C > 0 \), depending only on \( \alpha \) and \( K \) such that

\[
\varphi \left( \frac{|f_n(z)|}{M} \right) \leq CM_{\alpha, \varphi} \left( \frac{f_n}{M} \right) \leq C,
\]

for all \( n \in \mathbb{N}, z \in K, \) from this we have \( |f_n(z)| \leq M\varphi^{-1}(C) \) for all \( n \in \mathbb{N} \) and \( z \in K \). Which means that \( \{f_n\} \) is a uniformly bounded sequence of analytic functions on compact subsets of \( \mathbb{D} \). Montel’s Theorem (see [1]) guarantees that \( \{f_n\} \) is a normal family containing a subsequence \( \{f_{n_k}\} \) which uniformly convergent on compact subsets of \( \mathbb{D} \) to a function \( f \in H(\mathbb{D}) \). Hence \( \| f \|_{\alpha, \varphi} \leq M \), then \( M_{\alpha, \varphi} \left( \frac{f}{M} \right) \leq 1 \) and so \( f \in A_\alpha^\varphi \).

Moreover, the sequence \( \{f_{n_k} - f\} \) satisfy \( \| f_{n_k} - f \|_{\alpha, \varphi} \leq 2M \) and \( f_{n_k} - f \) converges uniformly to zero on compact subsets \( \mathbb{D} \). From this is easy to see that \( \| (f_{n_k} - f) \circ \phi \|_{\alpha, \varphi} \to 0 \) as \( k \to \infty \), then \( \{C_\phi(f_{n_k})\} \) converges in \( A_\alpha^\varphi \), thus \( C_\phi \) is a compact operator, and the proof is finished.

We now gives a necessary condition for the symbol \( \phi \) in order to the composition operator \( C_\phi \) be a compact on \( A_\alpha^\varphi \).
Theorem 3.3 If $\phi : \mathbb{D} \to \mathbb{D}$ is an analytic function such that
\[
\int_{\mathbb{D}} \frac{dA_{\alpha}}{(1 - |\phi|)^{2+\alpha}} < \infty, \tag{15}
\]
then $C_{\phi}$ is a compact operator on $A_{\alpha}^\varphi$.

Proof. Let $\{f_n\}$ be a bounded sequence in $A_{\alpha}^\varphi$ such that converges uniformly to zero on compact subsets of $\mathbb{D}$. We can assume, without lost of generality that $\|f_n\|_{\alpha,\varphi} \leq 1$ for all $n \in \mathbb{N}$, from this we have $M_{\alpha,\varphi}(f_n) \leq 1$ for all $n \in \mathbb{N}$. We claim that $\|C_{\phi}(f_n)\|_{\alpha,\varphi} \to 0$ as $n \to \infty$.

To this end, for a given $\varepsilon > 0$, we see that from condition (1) and (5) in Theorem 2.1, there exists a constant $C(\alpha) > 0$ such that
\[
\varphi \left( \frac{1}{\varepsilon} |f_n(w)| \right) \leq K_1 \left( \frac{1}{\varepsilon} \right) \varphi \left( |f_n(w)| \right) \leq \frac{K_1 \left( \frac{1}{\varepsilon} \right)}{C(\alpha) (1 - |w|)^{2+\alpha}},
\]
for all $n \in \mathbb{N}$, where we have used the fact that $M_{\alpha,\varphi}(f_n) \leq 1$. Taking $w = \phi(z)$, the latter inequality became
\[
\varphi \left( \frac{1}{\varepsilon} |f_n \circ \phi(z)| \right) \leq \frac{K_1 \left( \frac{1}{\varepsilon} \right)}{C(\alpha) (1 - |\phi(z)|)^{2+\alpha}}, \tag{16}
\]
for each $z \in \mathbb{D}$. By (15) there exists $r \in (0, 1)$ such that
\[
\int_{A_r} \frac{dA_{\alpha}}{(1 - |\phi|)^{2+\alpha}} < \frac{C(\alpha)}{2K_1 \left( \frac{1}{\varepsilon} \right)}, \tag{17}
\]
where $A_r = \{z \in \mathbb{D} : |z| > r\}$, thus integrating (16) on $A_r$ and applying (17) we have
\[
\int_{A_r} \varphi \left( \frac{1}{\varepsilon} |f_n \circ \phi(z)| \right) dA_{\alpha}(z) < \frac{1}{2}. \tag{18}
\]

On the other hand, since $\varphi$ is a continuous function, there is a $\bar{\varepsilon} > 0$ such that $\varphi(\bar{\varepsilon}) < \frac{1}{2}$. Also $D_r = \mathbb{D} \setminus A_r$ is a compact subset, then $\phi(D_r)$ is a compact subset of $\mathbb{D}$ because $\phi$ is a continuous function. From the fact $f_n \to 0$ on compact subsets of $\mathbb{D}$, we can find a $n_0 \in \mathbb{N}$ such that
\[
\sup_{z \in D_r} \frac{1}{\varepsilon} |f_n \circ \phi(z)| < \bar{\varepsilon}
\]
for all $n > n_0$, then $\varphi \left( \frac{1}{\varepsilon} |f_n \circ \phi(z)| \right) < \varphi(\bar{\varepsilon}) < \frac{1}{2}$ for all $z \in D_r$ and all $n > n_0$. Hence
\[
\int_{D_r} \varphi \left( \frac{1}{\varepsilon} |f_n \circ \phi(z)| \right) dA_{\alpha}(z) < \frac{1}{2}, \tag{19}
\]
whenever $n > n_0$. Finally, by (18) and (19) we have
\[
\int_D \varphi \left( \frac{1}{\varepsilon} |f_n \circ \phi| \right) \, dA_\alpha = \int_{D_r} \varphi \left( \frac{1}{\varepsilon} |f_n \circ \phi| \right) \, dA_\alpha + \int_{A_r} \varphi \left( \frac{1}{\varepsilon} |f_n \circ \phi| \right) \, dA_\alpha
\]
whenever $n > n_0$, therefore $\|C_\phi (f_n)\|_{\alpha, \varphi} \to 0$ as $n \to \infty$, and the proof is finished.

**ACKNOWLEDGEMENTS.** This research has been partially supported by the "Comisión de Investigación" of the "Universidad de Oriente", Venezuela.

**References**


**Received:** October 20, 2007