Characterisation of $L^1$-Multipliers for the Heisenberg Group

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Abstract

In this paper, a characterisation of Fourier multipliers for $L^1(H^n)$ and $L^1(H^n, A)$ are discussed, where $H^n$ denotes the Heisenberg group and $A$ is a commutative Banach algebra with a bounded approximate identity.

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1 Introduction

Characterisation of multiplier problem is one of the important and interesting problems in Harmonic Analysis. It is well known that the class of multipliers for $L^1(\mathbb{R}^n)$ coincides with the class of Fourier transforms of elements of finite Borel measures on $\mathbb{R}^n$. The class of multipliers for $L^2(\mathbb{R}^n)$ is the class of bounded measurable functions on $\mathbb{R}^n$. However for $p \neq 1, p \neq 2$, such a simple characterisation is not known. The classical theorem of Hörmander gives sufficient conditions for a Fourier multiplier for $L^p(\mathbb{R}^n)$, with $1 < p < \infty$.

In general for a locally compact abelian group $G$, the following theorem is well known.
Theorem 1.1. [12, Theorem 1.1] Let $G$ be a locally compact abelian group. Suppose $T : L^1(G) \to L^1(G)$ is a continuous linear transformation. Then the following conditions are equivalent:

1. $T$ commutes with translation operators, that is, $T\tau_s = \tau_s T$ for each $s \in G$.
2. $T(f * g) = Tf * g$ for each $f, g \in L^1(G)$.
3. There exists a unique function $\phi$ defined on $\hat{G}$ such that $(Tf)\hat{ } = \phi\hat{ } f$ for each $f \in L^1(G)$.
4. There exists a unique measure $\mu \in \mathcal{M}(G)$ such that $(Tf)\hat{ } = \hat{\mu} f$ for each $f \in L^1(G)$.
5. There exists a unique measure $\mu \in \mathcal{M}(G)$ such that $Tf = f * \mu$ for each $f \in L^1(G)$.

In fact, the equivalence of statements (1), (2), (5) in the above theorem were proved by Wendel in [23] for a general locally compact group. That theorem does not give a characterisation in terms of the Fourier transform of functions on $G$. However, the definition of Fourier transform depends on the representation theory of the group $G$ when $G$ is non abelian. In the case of locally compact abelian group $G$, one of the above five conditions in theorem 1.1 is used as a definition of multiplier for $L^1(G)$. However for $p \neq 1$ the statement (1) is used as the definition of multiplier for $L^p(G)$ in the literature. A necessary condition for $L^p(G)$ multiplier is given in terms of convolution with a pseudomeasure when $p \neq 1$. The class of multipliers for $L^p(G)$ is proved to be the dual space of a projective tensor product of $L^p(G)$ and $L^q(G)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$ in [7].

Hörmander’s theorem uses the following definition of Fourier multiplier: Let $m$ be a bounded measurable function on $\mathbb{R}^n$. For $f \in L^2 \cap L^p(\mathbb{R}^n), 1 \leq p < \infty$ define a linear transformation $T_m$ by $(T_m f)\hat{ } = m\hat{ } \hat{f}$ where $\hat{f}$ denotes the Fourier transform of $f$. If $T_m f \in L^p(\mathbb{R}^n)$ and $T_m$ is bounded, then we say that $m$ is multiplier for $L^p(\mathbb{R}^n)$. Further if $m$ is a multiplier for $L^p(\mathbb{R}^n)$, then $T_m$ is bounded on $L^p(\mathbb{R}^n)$ which commutes with translations. On the other hand if $T$ is a bounded linear transformation on $L^p(\mathbb{R}^n), p < \infty$, which commutes with translations then there exists a multiplier for $L^p(\mathbb{R}^n)$ such that $T_m = T$.

Hörmander’s multiplier theorem for the Heisenberg group was proved by Michele and Mauceri in [13]. The characterisation of right translation invariant operators in terms of convolution operators has been studied in a more general abstract sense for amenable groups in [6] and [8]. We also refer to Cowling [4] and Pier [14] for further details in this direction. Identification of convolution
operators for $L^p(G)$ with the dual space of projective tensor product of $L^p(G)$ and $L^q(G)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$ has been extended to a locally compact group by Cowling in [3]. Radha and Vijayarajan [17] has obtained a convolution type characterisation for Fourier $L^p$-multipliers for the Heisenberg group when $1 < p < \infty$.

The vector valued version of characterisation of the multipliers has been studied by various authors. Tewari, Dutta, and Vaidya [21] proved that the class of multipliers $M(L^1(G,A))$ is isometrically isomorphic to $\mathcal{M}(G,A)$, the class of $A$ valued bounded measures on $G$ where $G$ is a locally compact abelian group and $A$ is a commutative Banach algebra with identity. Later this work has been extended to multipliers for $L^1(G,A) \to L^p(G,A)$ and $L^1(G,A) \to L^p(G,X)$ where $X$ is an $A$-module using Radon-Nikodym property (see [10], [11], [15]) and without using RNP (see [16]).

In this paper we prove the analogue of theorem 1.1 for both complex valued and vector valued functions on the Heisenberg group and show that these statements coincide with the definition of multipliers which are defined through the Fourier transform of the Heisenberg group.

We organize our paper as follows. In section 2, we provide the notations and necessary background. In section 3, we discuss the multipliers for the complex valued functions on the Heisenberg group. In section 4 we discuss the vector version of the above result.

## 2 Preliminary Notes

Let $H^n$ denote the Heisenberg group. It is a unimodular nilpotent Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ and the group operation is defined by

$$(z,t) \cdot (w,s) = (z+w, t+s + \frac{1}{2} Im(z\bar{w})).$$

The Haar measure is given by $dzdt$.

By Stone-von Neumann theorem , the only infinite dimensional unitary irreducible representations (up to unitary equivalence) are given by $\pi_\lambda$, $\lambda \in \mathbb{R}^*$, where $\pi_\lambda$ is defined by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda (x\xi + \frac{1}{2} x y)} \varphi(\xi + y),$$

where $z = x + iy$ and $\varphi \in L^2(\mathbb{R}^n)$.

The group Fourier transform of $f \in L^1(H^n)$ is defined as

$$\hat{f}(\lambda) = \int_{H^n} f(z,t) \pi_\lambda(z,t) dzdt, \quad \lambda \in \mathbb{R}^*.$$

Notice that $\hat{f}(\lambda) \in \mathcal{B}(L^2(\mathbb{R}^n))$, bounded operators on $L^2(\mathbb{R}^n)$ and $||\hat{f}(\lambda)||_B \leq ||f||_{L^1(H^n)}$. As in the case of $\mathbb{R}^n$, the group Fourier transform $\hat{f}$ satisfies the
basic properties, namely if \( f \in L^1 \cap L^2(H^n) \), \( \hat{f}(\lambda) \) is a Hilbert-Schmidt operator. Further if we define \( d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda \), and if \( L^2(\mathbb{R}^*, \mathcal{B}_2, d\mu) \) denotes the collection of square integrable \( \mathcal{B}_2 \)-valued functions on \( \mathbb{R}^* \) under the measure \( d\mu \), then the group Fourier transform is an isometric isomorphism between \( L^2(H^n) \) and \( L^2(\mathbb{R}^*, \mathcal{B}_2, d\mu) \). Here \( \mathcal{B}_2 \) denotes the class of Hilbert-Schmidt operators on \( L^2(\mathbb{R}^n) \). The inversion formula is given by

\[
 f(z,t) = \int tr(\pi_\lambda(z,t)^*\hat{f}(\lambda)) \, d\mu(\lambda).
\]

If \( f, g \in L^1(H^n) \),

\[
 (f * g)(z,t) = \int f((z,t)(w,s)^{-1}) \, g(w,s) \, dwds
\]

denotes their convolution, then \( (f * g)^\wedge(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda) \). Under this convolution operation \( L^1(H^n) \) becomes a non-commutative algebra. For further results of the group Fourier transform we refer to Thangavelu [20].

3 Main Results

Definition 3.1. Let \( m \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu) \). Define for \( f \in L^2 \cap L^p(H^n) \), \( (T_m f)^\wedge = m \hat{f} \). If \( T_m f \in L^p(H^n) \) and \( T_m \) is a bounded operator, then we say that \( m \) is a multiplier for \( L^p(H^n) \).

Let \( A \) denote a commutative Banach algebra. \( L^1(H^n, A) \) denotes the space of all equivalence class of A-valued Bochner integrable functions defined on \( H^n \) with

\[
 \int_{H^n} \|f(s)\|_A ds < \infty
\]

and \( L^\infty(H^n, A) \), the space of all equivalence classes of A-valued Bochner integrable functions defined on \( H^n \) that are essentially bounded. \( \mathcal{M}(H^n, A) \) denotes the space of all bounded A-valued measures on \( H^n \). For each \( s \in H^n \) define \( (R_s f)(t) = f(ts) \) for all \( t \in H^n \).

We first observe the following.

Proposition 3.2. For \( f, g \in L^1(H^n) \), we have

1. \( f * R_s g = R_s(f * g) \)

2. \( (R_s f)^\wedge(\lambda) = \hat{f}(\lambda)\pi_\lambda(s^{-1}) \) for each \( s \in H^n \)

These follow from the definition of convolution, \( R_s, \hat{f}(\lambda) \) and applying suitable change of variables.

Now, we are in position to state our main result.
Theorem 3.3. Let $T$ be a bounded linear operator on $L^1(H^n)$. Then the following statements are equivalent:

1. $T$ commutes with right translation operators, that is, $TR_s = R_sT$ for each $s \in H^n$.
2. $T(f * g) = Tf * g$ for each $f, g \in L^1(H^n)$.
3. Then there exists a measure $\nu \in \mathcal{M}(H^n)$ such that $Tf = \nu * f$ for each $f \in L^1(H^n)$.
4. There exists a measure $\nu \in \mathcal{M}(H^n)$ such that $(Tf)\hat{\lambda} = \hat{\nu}f(\lambda)$ for each $f \in L^1(G)$.
5. There exists a function $\phi \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu)$ such that $(Tf)\hat{\phi}(\lambda) = \phi(\lambda)\hat{f}(\lambda)$ for each $f \in L^1(\mathbb{R}^*)$.

Proof. As mentioned earlier the statements (1), (2), (3) are proved to be equivalent in [23].

$(3) \Rightarrow (4)$. By applying Fourier transform on both sides of (3) we get (4).

$(4) \Rightarrow (5)$. Define $\hat{\nu} = \hat{\nu}$, where $\hat{\nu}(\lambda) = \int_{H^n} \pi_\lambda(x)d\nu(x)$. Then

$$||\phi(\lambda)||_B = ||\hat{\nu}(\lambda)|| \leq ||\pi_\lambda||_B||\nu||_{\mathcal{M}(H^n)} = ||\nu||_{\mathcal{M}(H^n)},$$

as $\pi_\lambda$ is a unitary representation of $H^n$ on $L^2(\mathbb{R}^n)$. Then $\phi \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu)$ and (5) follows. $(5) \Rightarrow (1)$. In order to prove $TR_s = R_sT$, consider for each $\lambda \in \mathbb{R}^*$ and $f \in L^1(H^n)$

$$(TR_s f)\hat{\phi}(\lambda) = \phi(\lambda)(R_s f)\hat{\phi}(\lambda) = \phi(\lambda)f(\lambda)\pi_\lambda(s^{-1}),$$

using proposition 3.2(2). On the other hand,

$$(R_s T f)\hat{\phi}(\lambda) = (T f)\hat{\lambda} \pi_\lambda(s^{-1}) = \phi(\lambda)f(\lambda)\pi_\lambda(s^{-1}).$$

By uniqueness of Fourier transform we get $TR_s f = R_sT f$ for each $f \in L^1(H^n)$ which in turn implies that $TR_s = R_sT$. \hfill \Box

Remark 3.4. Let $m$ be a multiplier for $L^1(H^n)$, viz, if we take $f \in L^1(H^n)$ and define $(T_m f)\hat{\phi} = m\hat{f}$, then $T_m$ is a bounded operator on $L^1(H^n)$. In other words if we assume condition (5) of theorem 3.3 then by (3) $\exists \nu \in \mathcal{M}(H^n)$ such that $T_m f = \nu * f$. Thus a multiplier on $L^1(H^n)$ leads to a convolution operator on $L^1(H^n)$ where the convolution is with respect to a measure $\nu \in \mathcal{M}(H^n)$.

Conversely if $T_m f = \nu * f \forall f \in L^1(H^n)$ and for some $\nu \in \mathcal{M}(H^n)$, then $||T_m f||_1 = ||\nu * f||_1 \leq ||\nu||_{\mathcal{M}(H^n)} ||f||_1$, which shows that $T_m$ is bounded operator on $L^1(H^n)$. Further we have by theorem 3.3, $(3) \Rightarrow (5)$ $\exists \phi \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu)$ such that $(T_m f)\hat{\phi} = \hat{f}$ proving that $m$ is a multiplier. Thus as in the case of locally compact abelian groups, a multiplier is equivalent to any of the conditions of theorem 3.3.
4 Vector Version of multipliers

In this section, we wish to obtain an analogue of theorem 3.3. Before stating the theorem we need to provide certain definitions.

**Definition 4.1.** Let $A$ be a commutative Banach algebra. For $x, y \in A, f \in A^*, F, G \in A^{**}$, we define the following:

\[(xof)(y) = f(yx), \quad \text{then} \quad xof \in A^*\]
\[(foF)(x) = F(xof), \quad \text{then} \quad foF \in A^*\]
\[(FoG)(f) = G(foF), \quad \text{then} \quad FoG \in A^{**}.\]

Then $A^{**}$ is a Banach algebra under the product ‘o’. This product is called Arens product. There is an isometric isomorphism of $(A, .)$ into $(A^{**}, o)$. We refer to [1], [22] for further details.

**Definition 4.2.** Let $A$ be a Banach algebra. Let $X$ be a Banach space. Then $X$ is said to be a Banach module over $A$ if $X$ is a module over $A$ in the algebraic sense and if it satisfies $\|ax\|_X \leq \|a\|_A \|x\|_X, a \in A, x \in X$.

**Example 4.3.** Trivially $\mathcal{B}(L^2(\mathbb{R}^n))$ is a Banach module over $\mathbb{C}$. We can show that $\mathcal{B}(L^2(\mathbb{R}^n))$ is a Banach module over $L^1(\mathbb{R}^n)$ under the following module operation:

For $f \in L^1(\mathbb{R}^n), T \in \mathcal{B}(L^2(\mathbb{R}^n))$

$$(fT)(g) = f \ast Tg \quad \text{for} \quad g \in L^2(\mathbb{R}^n).$$

Then $fT$ will be in $\mathcal{B}(L^2(\mathbb{R}^n))$ and it satisfies all algebraic properties of module over $L^1(\mathbb{R}^n)$.

Now

$$\|fT\|_B = \sup_g \{(fT)(g)\|_{L^2(\mathbb{R}^n)}/\|g\|_2 \leq 1\}$$

As

$$\|(fT)(g)\|_{L^2(\mathbb{R}^n)} = \|f \ast Tg\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|Tg\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|T\|_B \|g\|_{L^2(\mathbb{R}^n)}.$$

we get $\|fT\|_B \leq \|f\|_{L^1(\mathbb{R}^n)} \|T\|_B.$ Thus $\mathcal{B}(L^2(\mathbb{R}^n))$ is a Banach module over $L^1(\mathbb{R}^n)$.

**Example 4.4.** Let $D$ be a compact disk in $\mathbb{R}^n$. Let $A$ denote the Banach algebra of all complex valued continuous functions defined on $D$. For $f \in L^2(\mathbb{R}^n), T \in \mathcal{B}(L^2(\mathbb{R}^n))$, define

$$f \cdot T(g) = \begin{cases} f(t)Tg(t) & \text{for} \quad t \in D \\ 0 & \text{otherwise}. \end{cases}$$
Then \( f \cdot T \) will be in \( \mathcal{B}(L^2(\mathbb{R}^n)) \) and it satisfies all algebraic properties of module over \( A \).

Also

\[
\| (fT)(g) \|_{L^2(\mathbb{R}^n)} \leq \| f \|_{\infty} \| Tg \|_{L^2(\mathbb{R}^n)}.
\]

Thus \( \mathcal{B}(L^2(\mathbb{R}^n)) \) is a Banach module over \( A \).

**Definition 4.5.** For \( f \in L^1(H^n, A), g \in C_0(H^n, A^*) \) define for \( x \in H^n \),

\[
f \odot g(x) = \int_{H^n} f(xy^{-1})og(y)dy.
\]

where ‘\( \odot \)’ is defined in equation (1) above.

Clearly \( \| f \odot g(x) \|_{A^*} \leq \| f \|_{L^1(H^n, A)} \| g \|_{C_0(H^n, A^*)} \), which shows that the definition is meaningful and \( f \odot g(x) \in C_0(H^n, A^*) \).

**Definition 4.6.** If \( f \in L^1(H^n, A), \nu \in \mathcal{M}(H^n, A^{**}) \) define for each \( h \in C_0(H^n, A^*) \), \( \nu \star f \) by

\[
\langle h, \nu \star f \rangle = \langle \tilde{f} \odot h, \nu \rangle = \nu(\tilde{f} \odot h)
\]

where \( \tilde{f}(x) = f(x^{-1}) \).

The above definition is meaningful since \( \mathcal{M}(H^n, A^{**}) \) is the dual space of \( C_0(H^n, A^*) \) and the duality relation can be used to show that \( \nu \star f \in \mathcal{M}(H^n, A^{**}) \) and in particular if \( \nu \in \mathcal{M}(H^n, A) \) (notice that \( \mathcal{M}(H^n, A) \) is a subspace of \( \mathcal{M}(H^n, A^{**}) \) then \( \nu \star f = \nu \star f \).

**Definition 4.7.** Let \( m \in L^\infty(\mathbb{R}^*, \mathcal{B}, d\mu) \). Define for \( f \in L^1(H^n, A) \), \((T_m f)^\wedge = m \hat{f}\). If \( T_m f \in L^1(H^n, A) \) and \( T_m \) is a bounded operator, then we say that \( m \) is a multiplier for \( L^1(H^n, A) \).

Given two Banach spaces \( X \) and \( Y \), the projective tensor product of \( X \) and \( Y \) is denoted by \( X \otimes Y \) (see[5]). If \( X_i, Y_i \) \( (i = 1, 2) \) are Banach spaces and \( T_i : X_i \rightarrow Y_i \) \( (i = 1, 2) \) are bounded linear operators then the equation

\[
T_1 \otimes T_2(x_1 \otimes x_2) = T_1(x_1) \otimes T_2(x_2)
\]

defines a bounded linear operator from \( X_1 \otimes X_2 \) to \( Y_1 \otimes Y_2 \) (see[18]).

In the following theorems we will obtain the equivalent conditions for the multiplier.

**Theorem 4.8.** Let \( A \) be a commutative Banach algebra with bounded approximate identity. Let \( T : L^1(H^n, A) \rightarrow L^1(H^n, A) \) be a continuous linear operator. Then the following conditions are equivalent:
1. $T(\lambda_a \otimes R_s) = (\lambda_a \otimes R_s T)$ for each $a \in A, s \in H^n$ where $\lambda_a$ is the operator on $A$ given by $\lambda_a(b) = ab$ and $R_s$ is the right translation operator.

2. $TR_s = R_s T$, $s \in H^n$ and $T(xf) = xTf \ \forall x \in A, f \in L^1(H^n, A)$.

Proof. It can be easily verified that $R_s(f \ast g) = f \ast R_s g \ \forall f \in L^1(H^n), g \in L^1(H^n, A)$ for each $s \in H^n$. By returning to the proof of proposition 2.5 in [16], we obtain the required result, which gives an equivalent statement for the multiplier.

Now, we prove the analogue of theorem 3.3.

**Theorem 4.9.** Let $A$ be a commutative Banach Algebra with bounded approximate identity. Let $T : L^1(H^n, A) \to L^1(H^n, A)$ be a bounded linear operator. Then the following conditions are equivalent:

1. $TR_sf = R_s T f$, $T(xf) = xTf \ \forall x \in A, f \in L^1(H^n, A)$ for each $s \in H^n$.

2. $T(f \ast g) = T f \ast g$ for each $f, g \in L^1(H^n, A)$.

3. There exists a measure $\nu \in \mathcal{M}(H^n, A^{**})$ such that $Tf = \nu \ast f$ for each $f \in L^1(H^n, A)$.

In addition if $A$ is a reflexive Banach space over which $\mathcal{B}(L^2(\mathbb{R}^n))$ is a module, then the above conditions are equivalent to the following:

4. There exists a measure $\nu \in \mathcal{M}(H^n, A)$ such that $(Tf) = \hat{\nu}f$ for each $f \in L^1(H^n, A)$.

5. There exists a function $\phi \in L^\infty(\mathbb{R}^n, \mathcal{B}, d\mu)$ such that $(Tf)(\lambda) = \phi(\lambda)\hat{f}(\lambda)$ for each $f \in L^1(H^n, A)$.

Proof. If the function $t \mapsto \langle f(t), g(t) \rangle$ is negligible for every $g \in L^\infty(H^n, A^*)$, then $f \in L^1(H^n, A)$ is negligible as the functions $x^*\phi$ ($x^* \in A^*$, $\phi \in C_0(H^n)$) belong to $L^\infty(H^n, A^*)$ see Bourbaki[2]. By rewriting the proof of (1) $\Rightarrow$ (2) in [16], we can show that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) Let $\{g_\alpha\}$ be an approximate identity for $L^1(H^n, A)$ such that $\|g_\alpha\|_{L^1(H^n, A)} = 1$. Consider $\|Tg_\alpha*f - Tf\|_{L^1(H^n, A)} = \|T(g_\alpha*f) - Tf\|_{L^1(H^n, A)} \leq \|T\| \|g_\alpha*f-f\|_{L^1(H^n, A)} \to 0$ as $\alpha \to \infty$. As $\|Tg_\alpha\|_{L^1(H^n, A)} \leq \|T\| \|g_\alpha\|_{L^1(H^n, A)} = \|T\|$, $\{Tg_\alpha\}$ is a bounded subset of $L^1(H^n, A)$. Since $L^1(H^n, A) \subset \mathcal{M}(H^n, A)$ and

$$\|f\|_{\mathcal{M}(H^n, A)} \leq \|f\|_{L^1(H^n, A)}, \forall f \in L^1(H^n, A) \quad (2)$$

we have $\{Tg_\alpha\}$ is a norm bounded subset of $\mathcal{M}(H^n, A)$. But $\mathcal{M}(H^n, A) \subset \mathcal{M}(H^n, A^{**})$ which is the dual space of $C_0(H^n, A^*)$. By applying Banach - Alaoglu’s theorem there is a $\nu \in \mathcal{M}(H^n, A^{**})$ such that $Tg_\beta \to \nu$ in the weak* topology and also

$$\langle Tf, g \rangle = \lim_{\beta} \langle Tg_\beta * f, g \rangle \ \forall f \in C_0(H^n, A), g \in C_0(H^n, A^*)$$
By definition we have
\[ \langle h, Tg_\beta \ast f \rangle = Tg_\beta(\hat{f} \otimes h) \]
for \( h \in \mathcal{C}_C(H^n, A^*) \). But
\[ \lim_{\beta \to \infty} Tg_\beta(\hat{f} \otimes h) = \nu(\hat{f} \otimes h) = \langle h, \nu \otimes f \rangle. \]
Thus
\[ \lim_{\beta \to \infty} \langle h, Tg_\beta \ast f \rangle = \langle h, \nu \otimes f \rangle \]
for all \( h \in \mathcal{C}_C(H^n, A^*) \). But since \( Tg_\beta \in L^1(H^n, A) \subset M(H^n, A^*) \), \( Tg_\beta \ast f = Tg_\beta \ast f \) weakly, we conclude that \( Tf = \nu \ast f \) as elements of \( M(H^n, A^*) \). Since \( \mathcal{C}_C(H^n, A^*) \) is dense in \( C_0(H^n, A^*) \), \( Tf \in L^1(H^n, A) \) and \( L^1(H^n, A) \subset M(H^n, A^*) \) we have \( Tf = \nu \ast f \) for each \( f \in L^1(H^n, A) \).

Proof of (3) \( \Rightarrow \) (1) is similar to that of the proof of (4) \( \Rightarrow \) (1) in [16]. Since \( A \) is reflexive \( A = A^{**} \). Then by (3) there exists a \( \nu \in M(H^n, A) \) such that \( Tf = \nu \ast f = \nu \ast f \) for each \( f \in L^1(H^n, A) \) which in turn implies \( (Tf) = \nu \hat{f} \) for each \( f \in L^1(H^n, A) \), thus proving (3) \( \Rightarrow \) (4). Then \( \phi = \hat{\nu} \) gives (4) \( \Rightarrow \) (5).

Now we shall prove (5) \( \Rightarrow \) (2). Consider \( [T(f \ast g)] = \phi(f \ast g) = \phi \hat{f} \hat{g} = (Tf) \hat{g} = (Tf \ast g) \). Then the result follows by the uniqueness of Fourier transform.

\textbf{Remark 4.10.} As discussed in remark 3.4, we can show that the definition of multiplier (vector valued) is equivalent to the statements in theorem 4.7 and 4.8.

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\section*{References}


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