Estimates for Boundary Blow-up Solutions of Semilinear Elliptic Equations

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Abstract

We investigate boundary blow-up solutions of the equation $\Delta u = f(u)$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$. Under the condition that $f(t)$ grows exponentially as $t$ goes to infinity we show how the mean curvature of the boundary $\partial \Omega$ appears in the asymptotic expansion of the solution $u(x)$ in terms of the distance of $x$ from the boundary $\partial \Omega$.

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1 Introduction and assumptions

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth non negative increasing function which satisfies the Keller-Osserman condition

$$\int_1^\infty \frac{dt}{(2F(t))^{\frac{1}{2}}} < \infty, \quad F'(t) = f(t). \quad (1)$$

We also suppose that the integral of $f(t)$ over $(-\infty, 0)$ is finite. It is well known [10], [12] that under these conditions the Dirichlet problem

$$\Delta u = f(u) \text{ in } \Omega, \quad u(x) \rightarrow \infty \text{ as } x \rightarrow \partial \Omega, \quad (2)$$

has a classical solution called boundary blow-up (explosive, large) solution. Moreover, the one dimensional problem

$$\phi'' = f(\phi), \quad \phi'(s) > 0, \quad \lim_{s \rightarrow 0} \phi(s) = \infty$$
has a solution satisfying
\[ \int_{\phi(s)}^{\infty} \frac{dt}{(2F(t))^{1/2}} = s, \quad F(t) = \int_{-\infty}^{t} f(\tau)d\tau. \] (3)

Under some additional condition on \( f \), it is possible to show the estimate [7]
\[ \lim_{x \to \partial \Omega} \frac{u(x)}{\phi(\delta(x))} = 1, \]
where \( \delta(x) \) denotes the distance of \( x \) from \( \partial \Omega \). This means that the main part of the asymptotic expansion of the solution \( u(x) \) in terms of \( \delta(x) \) is independent of the geometry of the domain. The behaviour of large solutions near the boundary has been investigated by many researchers, see for example, [2], [5], [6], [9], [11]. Let us recall a result of C. Bandle and M. Marcus, taken from Theorem 4 of [7].

**Lemma 1.1 (Bandle-Marcus)** Let the Keller-Osserman condition (1) hold and let \( F(t)/t^2 \) be increasing for large \( t \). Moreover, if \( G(t) = \int_{0}^{t} (F(\tau))^{1/2} d\tau \), suppose there exist \( a, b \), with \( 1 < a < b \) such that
\[ a \frac{F(t)}{f(t)} \leq \frac{G(t)}{G'(t)} \leq b \frac{F(t)}{f(t)} \] (4)
for large \( t \). Then
\[ M \frac{(\delta(x))^2}{\phi(\delta(x))} \frac{\phi'(\delta(x))}{\phi(\delta(x))} \leq \frac{u(x)}{\phi(\delta(x)))} - 1 \leq M \delta(x), \] (5)
where \( \phi \) is defined by (3) and \( M \) is a suitable constant.

The estimate (5) has been improved in [3] for a class of functions \( f(t) \) having a polynomial growth. For functions having an exponential growth, only particular cases have been investigated. C. Bandle, in [4], has discussed the case \( f(t) = e^t \), proving the expansion
\[ u(x) = \log \frac{2}{\delta^2} + (N - 1)K(\pi)\delta + o(\delta), \] (6)
where \( K(\pi) \) denotes the mean curvature of \( \partial \Omega \) at the nearest point to \( x \), and \( o(\delta) \) has the usual meaning. For the case \( f(t) = e^{t|t|\beta - 1} \), \( \beta > 0, \beta \neq 1 \), in [1] it is proved that
\[ u(x) = \phi(\delta)[1 + \beta^{-1}(N - 1)K(x)(\phi(\delta))^{-\beta}\delta + O(1)(\phi(\delta))^{-2\beta}\delta], \] (7)
where \( K(x) \) denotes the mean curvature of the surface \( \{ x \in \Omega : \delta(x) = \text{constant} \} \), \( O(1) \) (here and in what follows) denotes a bounded quantity, and \( \phi(s) \) is defined as in (3) using the function \( f(t) = e^{t|t|\beta - 1} \).
In the present paper we consider general classes of functions \( f(t) \) including the previous cases. More precisely, suppose \( f : \mathbb{R} \to \mathbb{R} \) is a \( C^2 \) function such that

\[
f(t) \geq 0, \quad F(t) = \int_{-\infty}^{t} f(\tau)d\tau < \infty; \quad f'(t) > 0 \quad \text{for } t \text{ large.} \quad (8)
\]

Moreover suppose that, for \( t \) large,

\[
\frac{F(t)f'(t)}{(f(t))^2} = 1 + O(1)\frac{1}{t}, \quad (9)
\]

and that there is a constant \( C > 0 \) such that

\[
\frac{F(t)}{f(t)} = C + O(1)\frac{1}{t}. \quad (10)
\]

Equation (10) implies, for \( t \) large,

\[ e^{c_1 t} < F(t) < e^{c_2 t}, \]

for some constants \( 0 < c_1 \leq c_2 \). As a consequence, the Keller-Osserman condition (1) holds. Furthermore, (9) and (10) imply

\[
\frac{f'(t)}{f(t)} = \frac{1}{C} + O(1)\frac{1}{t}. \quad (11)
\]

We also note that (10) and (11) imply (9).

We also need a condition on the growth of \( f''(t) \). Suppose that for some \( m > 2 \) there are \( \epsilon > 0, \ L > 0 \) and \( t_0 > 0 \) such that for all \( t > t_0 \) we have

\[
\frac{|f''(t + \sigma)|}{f(t)} \leq L(F(t))^{1/m}, \quad -\epsilon < \sigma < \epsilon. \quad (12)
\]

Under these conditions we shall prove that a solution \( u(x) \) to problem (2) satisfies

\[
u(x) = \phi(\delta) + (N - 1)K(x)C\delta + O(1)\frac{1}{\phi(\delta)}\delta, \quad (13)
\]

where \( \phi(s) \) is defined as in (3) (obviously using the present function \( f(t) \)).

Estimate (13) implies (6) when \( f(t) = e^t \). Other functions which satisfy (8), (9), (10) and (12) are \( f(t) = e^tP(t) \), where \( P(t) \) is any positive polynomial.

We also consider functions \( f(t) \) which satisfy conditions alternative to (9), (10) and (12). Define a function \( g(t) \) such that, for \( t > 1 \) we have

\[
g(t) > 0, \quad \lim_{t \to \infty} g'(t) = 0, \quad (14)
\]
and either
\[ \exists D > 0, \ t_0 > 0 : \frac{g(t)}{t|g'(t)|} < D, \ \frac{|g''(t)|}{|g'(t)|} < D, \ \frac{t^2|g'''(t)|}{|g'(t)|} < D \ \forall t > t_0, \tag{15} \]
or
\[ \exists D > 0, \ t_0 > 0 : \frac{g(t)}{|g'(t)|} < D, \ \frac{|g''(t)|}{|g'(t)|} < D, \ \frac{|g'''(t)|}{|g'(t)|} < D \ \forall t > t_0. \tag{16} \]

We suppose
\[ \frac{F(t)f'(t)}{(f(t))^2} = 1 + O(1)g'(t), \tag{17} \]
and
\[ \frac{F(t)}{f(t)} = g(t)(1 + O(1)g'(t)). \tag{18} \]

Observe that (17) and (18) imply
\[ \frac{f''(t)}{f(t)} = \frac{1}{g(t)}(1 + O(1)g'(t)). \tag{19} \]

We also note that (19) and (18) imply (17).

In addition, we suppose that for some \( m > 2 \) there are \( \epsilon > 0, \ L > 0 \) and \( t_0 > 0 \) such that for all \( t > t_0 \) we have
\[ \frac{|f''(t + \sigma)|}{f(t)} \leq L(g(t))^{-2}(F(t))^\frac{1}{m}, \quad -\epsilon < \sigma < \epsilon. \tag{20} \]

Under the above conditions we shall prove that a solution to problem (2) satisfies
\[ u(x) = \phi(\delta) + (N - 1)K(x)g(\phi(\delta))\delta + O(1)g(\phi(\delta))g'(\phi(\delta))\delta. \tag{21} \]

We present here two examples.

**Example 1.** For \( \beta > 0, \ \beta \neq 1 \) consider the function \( f(t) = e^{t|t|^{\beta-1}}P(t) \), where \( P(t) \) is a positive polynomial. Let us prove that it satisfies conditions (18), (19) and (20) with \( g(t) = \frac{1}{\beta}t^{1-\beta} \). The function \( g(t) \) satisfies (14) and (15). Note that
\[ \lim_{t \to \infty} \frac{tP'(t)}{P(t)} = n, \tag{22} \]
where \( n \) is the degree of \( P(t) \). For \( t > 1 \) we have
\[ F(t) = C_1 + \int_1^t \frac{1}{\beta}\tau^{1-\beta}P(\tau)(\beta\tau^{\beta-1}e^{\tau^{\beta}})d\tau \]
Therefore,

\[ C_2 + \frac{1}{\beta} t^{1-\beta} e^\beta P(t) - \frac{1}{\beta} \int_1^t e^{\beta \tau}(\beta - 1)\tau^{-\beta} P(\tau) + \tau^{-\beta} P'(\tau) d\tau. \]

Hence,

\[
F(t) = \frac{C_2}{e^{\beta P(t)}} + \frac{1}{\beta} t^{1-\beta} + \frac{1}{\beta} \int_1^t e^{\beta \tau}(\beta - 1)\tau^{-\beta} P(\tau) - \tau^{-\beta} P'(\tau) d\tau.
\]  

(23)

Let \( \beta - 1 \neq n \). Using de l'Hôpital rule and (22) we find

\[
\lim_{t \to \infty} \frac{\int_1^t e^{\beta \tau}(\beta - 1)\tau^{-\beta} P(\tau) - \tau^{-\beta} P'(\tau) d\tau}{t^{1-2\beta} e^{\beta P(t)}} = \frac{(\beta - 1)P(t) - tP'(t)}{(1 - 2\beta)t^{-\beta} + \beta P(t) + t^{-\beta} tP'(t)} = \frac{\beta - 1 - n}{\beta}.
\]

One finds easily that this result holds also when \( \beta - 1 = n \). This estimate and (23) yield

\[
F(t) = \frac{C_2}{e^{\beta P(t)}} + \frac{1}{\beta} t^{1-\beta} + O(1) t^{1-2\beta},
\]

from which we get (18) with \( g(t) = \frac{1}{\beta} t^{1-\beta} \).

Let us prove (19). For \( t > 0 \) we find

\[
\frac{f'(t)}{f(t)} = \frac{P'(t)}{P(t)} + \beta t^{\beta-1},
\]

from which the result follows. As already observed, (17) follows from (18) and (19).

Let us show that also condition (20) holds. Since for \( t \) large

\[
\left| \frac{t P''(t)}{P'(t)} \right| \leq A,
\]

with some \( A \in \mathbb{R} \), we find

\[
f''(t) = f(t) \left[ \beta^2 t^{2\beta-2} + \beta(\beta - 1) t^{\beta-2} + 2\beta t^{\beta-1} \frac{P'(t)}{P(t)} + \frac{P''(t)}{P(t)} \right] \leq f(t) M_1 t^{2\beta-2}.
\]

Therefore,

\[
\left| \frac{f''(t + \sigma)}{f(t)} \right| \leq M_1 \frac{f(t + \sigma)}{f(t)} (t + \sigma)^{2\beta-2} = M_1 \frac{e^{(t + \sigma)^\beta} P(t + \sigma)}{e^{\beta P(t)}} (t + \sigma)^{2\beta-2}.
\]  

(24)

Since for \( t \) large we have

\[
\frac{e^{(t + \sigma)^\beta}}{e^{\beta} \left( \int_0^t e^{\beta \tau} P(\tau) d\tau \right)^{\frac{1}{\beta}}} \leq M_2,
\]
inequality (20) follows from (24).

Observe that (21) becomes (7) when \( f(t) = e^{\beta t} |t|^{-1} \), \( \beta \neq 1 \).

**Example 2.** Consider the function \( f(t) = e^{t + e^t} P(t) \), where \( P(t) \) is a positive polynomial. Let us prove that it satisfies conditions (18), (19) and (20) with \( g(t) = e^{-t} \). The function \( g(t) \) satisfies (14) and (16). We have

\[
F(t) = \int_{-\infty}^{t} (1 + e^{\tau}) e^{\tau + e^{\tau}} \frac{P(\tau)}{1 + e^{\tau}} d\tau
\]

\[
= e^{t + e^t} \frac{P(t)}{1 + e^t} - \int_{-\infty}^{t} e^{\tau + e^{\tau}} \left( \frac{P(\tau)}{1 + e^{\tau}} \right)' d\tau.
\]

Hence,

\[
\frac{F(t)}{f(t)} = \frac{1}{1 + e^t} - \frac{\int_{-\infty}^{t} e^{\tau + e^{\tau}} \left( \frac{P(\tau)}{1 + e^{\tau}} \right)' d\tau}{e^{t + e^t} P(t)}.
\]

We have

\[
\frac{1}{1 + e^t} = e^{-t} + O(1)e^{-2t}.
\]

Moreover, using de l'Hôpital's rule we find

\[
-\lim_{t \to \infty} \frac{\int_{-\infty}^{t} e^{\tau + e^{\tau}} \left( \frac{P(\tau)}{1 + e^{\tau}} \right)' d\tau}{e^{-2t} e^{t + e^t} P(t)} = 1.
\]

Hence, estimate (18) follows by (51)

Since

\[
f'(t) = e^{t + e^t} [(1 + e^t)P(t) + P'(t)],
\]

(19) follows easily.

Let us show that also condition (20) holds. For \( t \) large we have

\[
|f''(t)| \leq M_1 e^{2t} e^{t + e^t} P(t).
\]

Therefore,

\[
\frac{|f''(t + \sigma)|}{f(t)} \leq M_2 e^{2t} \frac{e^{t + e^t} P(t + \sigma)}{e^{t + e^t} P(t)}.
\]

(26)

Since

\[
\frac{e^{t + e^t}}{e^{t} \left( \int_{-\infty}^{t} e^{\tau + e^{\tau}} P(\tau) d\tau \right)_{\frac{1}{m}}} \leq M_3,
\]

provided \( e^\sigma - 1 < 1/m \), condition (20) follows from (26).
2 Main results

Lemma 2.1 Let \( \phi = \phi(s) \) be defined as in (3).
a) If \( f \) satisfies (8), (9) and (10) then
\[
-\frac{\phi'(s)}{sf(\phi(s))} = 1 + O(1) \frac{1}{\phi(s)}.
\]
(27)
b) If \( f \) satisfies (8), (17) and (18), with \( g(t) \) satisfying (14) and either (15) or (16) then
\[
-\frac{\phi'(s)}{sf(\phi(s))} = 1 + O(1)g'(\phi(s)).
\]
(28)

Proof. a) Inserting (9) into
\[
-1 + 2\left(1 + O(1)\frac{1}{t}\right) = 1 + O(1)\frac{1}{t}
\]
we have
\[
-1 + 2F(t)f'(t)(f(t))^{-2} = 1 + O(1)\frac{1}{t}.
\]
Multiplying by \((2F(t))^{-\frac{1}{2}}\) we find
\[
-(2F(t))^{-\frac{1}{2}} + (2F(t))^\frac{1}{2}f'(t)(f(t))^{-2} = (2F(t))^{-\frac{1}{2}} + O(1)(2F(t))^{-\frac{1}{2}}\frac{1}{t},
\]
and
\[
-((2F(t))^\frac{1}{2}(f(t))^{-1})' = (2F(t))^{-\frac{1}{2}} + O(1)(2F(t))^{-\frac{1}{2}}\frac{1}{t}.
\]
(29)
Condition (10) implies
\[
\lim_{t \to \infty} \frac{(F(t))^{\frac{1}{2}}}{f(t)} = C \lim_{t \to \infty} \frac{1}{(F(t))^{\frac{1}{2}}} = 0.
\]
(30)
Hence, integrating on \((t, \infty)\) in (2.3) we get
\[
(2F(t))^\frac{1}{2}(f(t))^{-1} = \int_t^\infty (2F(\tau))^{-\frac{1}{2}} d\tau + O(1) \int_t^\infty (2F(\tau))^{-\frac{1}{2}} \frac{1}{\tau} d\tau.
\]
(31)
Since
\[
\int_t^\infty (2F(\tau))^{-\frac{1}{2}} \frac{1}{\tau} d\tau < \frac{1}{t} \int_t^\infty (2F(\tau))^{-\frac{1}{2}} d\tau,
\]
by (2.5) we have
\[
(2F(t))^\frac{1}{2}(f(t))^{-1} = \int_t^\infty (2F(\tau))^{-\frac{1}{2}} d\tau + O(1) \frac{1}{t} \int_t^\infty (2F(\tau))^{-\frac{1}{2}} d\tau.
\]
Putting \( t = \phi(s) \) and using the equation \(-\phi'(s) = (2F(\phi(s)))^{\frac{1}{2}}\), the estimate (27) follows.

b) Inserting (17) into

\[-1 + 2(1 + O(1)g'(t)) = 1 + O(1)g'(t)\]

we have

\[-1 + 2F(t)f''(t)(f(t))^{-2} = 1 + O(1)g'(t).\]

Multiplying by \((2F(t))^{-\frac{1}{2}}\) we find

\[-(2F(t))^{-\frac{1}{2}} + (2F(t))^{\frac{3}{2}}f'(t)(f(t))^{-2} = (2F(t))^{-\frac{1}{2}} + O(1)(2F(t))^{-\frac{1}{2}}g'(t),\]

and

\[-((2F(t))^{\frac{3}{2}}f(t))^{-1}' = (2F(t))^{-\frac{1}{2}} + O(1)(2F(t))^{-\frac{1}{2}}g'(t). \quad (32)\]

By using (14) and either (15) or (16) we get

\[\lim_{t \to \infty} \frac{g(t)}{t} = 0.\]

Hence, (18) implies

\[\lim_{t \to \infty} \frac{F(t)}{tf(t)} = 0. \quad (33)\]

As a consequence, given \( p > 2 \) we have, for \( t \) large, \( F(t) > t^p \). Using this fact and (33) we find

\[\lim_{t \to \infty} \left( \frac{F(t)}{f(t)} \right)^{\frac{3}{2}} = \lim_{t \to \infty} \frac{F(t)}{tf(t)} \frac{t}{(F(t))^{\frac{1}{2}}} = 0. \quad (34)\]

Hence, integrating on \((t, \infty)\) in (2.6) we get

\[ (2F(t))^{\frac{3}{2}}(f(t))^{-1} = \int_t^\infty (2F(\tau))^{-\frac{1}{2}}d\tau + O(1)\int_t^\infty (2F(\tau))^{-\frac{1}{2}}g'(\tau)d\tau. \quad (35)\]

We have

\[
\lim_{t \to \infty} \frac{\int_t^\infty (2F(\tau))^{-\frac{1}{2}}g'(\tau)d\tau}{g'(t)\int_t^\infty (2F(\tau))^{-\frac{1}{2}}d\tau} = \lim_{t \to \infty} \frac{-(2F(t))^{-\frac{1}{2}}g'(t)}{g''(t)\int_t^\infty (2F(\tau))^{-\frac{1}{2}}d\tau - g'(t)(2F(t))^{-\frac{1}{2}}}
\]

\[= \lim_{t \to \infty} \frac{1}{-\frac{(2F(t))^{-\frac{1}{2}}g'(t)}{g''(t)\int_t^\infty (2F(\tau))^{-\frac{1}{2}}d\tau} + 1}. \quad (36)\]

If condition (15) holds then

\[0 \leq \lim_{t \to \infty} \left| \frac{g''(t)}{g'(t)} \right| \int_t^\infty (2F(\tau))^{-\frac{1}{2}}d\tau \leq \lim_{t \to \infty} D\int_t^\infty (2F(\tau))^{-\frac{1}{2}}d\tau \]

\[\leq \lim_{t \to \infty} \frac{\int_t^\infty (2F(\tau))^{-\frac{1}{2}}d\tau}{t(2F(t))^{-\frac{1}{2}}} \]
where (33) has been used in the last step. If condition (16) holds then \( g(t) \to 0 \) as \( t \to \infty \); as a consequence, (18) in this case implies that

\[
\lim_{t \to \infty} \frac{F(t)}{f(t)} = 0,
\]

and we have

\[
0 \leq \lim_{t \to \infty} \frac{|g''(t)|}{|g'(t)|} \int_t^\infty (2F(\tau))^{-\frac{1}{2}} d\tau \leq \lim_{t \to \infty} D \int_t^\infty (2F(\tau))^{-\frac{1}{2}} d\tau
\]

\[
= D \lim_{t \to \infty} \frac{(2F(t))^{-\frac{1}{2}}}{(2F(t))^{-\frac{1}{2}} f(t)} = D \lim_{t \to \infty} \frac{2F(t)}{f(t)} = 0.
\]

Hence, in both cases, (36) implies

\[
\lim_{t \to \infty} \frac{\int_t^\infty (2F(\tau))^{-\frac{1}{2}} g'(\tau) d\tau}{\int_t^\infty (2F(\tau))^{-\frac{1}{2}} d\tau} = 1,
\]

and by (35) we have

\[
(2F(t))^{\frac{1}{2}} (f(t))^{-1} = \int_t^\infty (2F(\tau))^{-\frac{1}{2}} d\tau + O(1) g'(t) \int_t^\infty (2F(\tau))^{-\frac{1}{2}} d\tau.
\]

Putting \( t = \phi(s) \) and using the equation \( -\phi'(s) = (2F(\phi(s)))^{\frac{1}{2}} \), the estimate (28) follows, and the lemma is proved.

**Theorem 2.2** Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^N \), \( N \geq 2 \), let \( \phi = \phi(s) \) be defined as in (3) and let \( u(x) \) be a solution of problem (2).

a) If \( f \) satisfies (8), (9), (10) and (12) then

\[
u(x) = \phi(\delta) + (N - 1)K(x)C\delta + O(1) \frac{1}{\phi(\delta)} \delta.
\]

b) If \( f \) satisfies (8), (17), (18) and (20), with \( g(t) \) satisfying (14) and either (15) or (16), then

\[
u(x) = \phi(\delta) + (N - 1)K(x)g(\phi(\delta))\delta + O(1)g(\phi(\delta))g'(\phi(\delta))\delta.
\]

Here \( \delta = \delta(x) \) is the distance from \( x \) to \( \partial \Omega \), \( O(1) \) denotes a bounded quantity, and \( K = K(x) \) is the mean curvature of the surface \{ \( x \in \Omega : \delta(x) = \text{constant} \) \}. 

Proof. a) We look for a super-solution of the form

\[ w(x) = \phi(\delta) + HC\delta + \alpha\delta(\delta(\delta))^{-1}, \]

where \( H = (N - 1)K \) and \( \alpha \) is a positive constant to be determined. We have

\[ w_{x_i} = \phi'\delta_{x_i} + H_{x_i}\delta + HC\delta_{x_i} + \alpha[\delta_{x_i}\phi^{-1} - \delta\phi^{-2}\phi'\delta_{x_i}]. \]

Using the equations

\[ \sum_{i=1}^{N} \delta_{x_i}\delta_{x_i} = 1, \quad \sum_{i=1}^{N} \delta_{x_i} = 0, \]

we find

\[
\Delta w = \phi'' - \phi' H + \Delta HC\delta + 2\nabla H \cdot \nabla \delta C - H^2 C
+ \alpha[-\phi^{-1} H - 2\phi^{-2}\phi' + 2\delta\phi^{-3}(\phi')^2 - \delta\phi^{-2}\phi'' + \delta\phi^{-2}\phi' H]. \tag{39}
\]

Equation (27) with \( s = \delta \) yields

\[ -\phi' = [1 + O(1)\phi^{-1}]\delta f(\phi). \tag{40} \]

Since \( \phi'' = f(\phi) \), by (39) and (40) we find

\[
\Delta w = f(\phi) \left\{ 1 + H_\delta + O(1)\frac{\delta}{\phi} + O(1)\frac{1}{f(\phi)} \right. \\
+ \left. \alpha \left[ O(1)\frac{1}{\phi f(\phi)} + O(1)\frac{\delta}{\phi^2} + O(1)\frac{\delta^2 f(\phi)}{\phi^3} \right] \right\}. \tag{41}
\]

As already observed, (10) implies that \( F(t) \) has an exponential growth. We claim that, for \( \delta \) small,

\[ \frac{1}{f(\phi)} \leq \frac{\delta}{\phi}. \tag{42} \]

Indeed, rewrite (42) as

\[ \frac{\phi(f(\phi))^{-1}}{\delta} \leq 1. \]

If \( \psi \) is the inverse function of \( \phi \), the latter inequality follows by the following result

\[
\lim_{\delta \to 0} \frac{\phi(f(\phi))^{-1}}{\delta} = \lim_{t \to \infty} \frac{F(t)}{\psi(t)} = \lim_{t \to \infty} \frac{(f(t))^{-1} - t(f(t))^{-2}f'(t)}{-(2F(t))^{-\frac{3}{2}}} \\
= \lim_{t \to \infty} \left[ -\sqrt{2} \frac{F(t)}{f(t)} \frac{1}{(F(t))^{\frac{3}{2}}} + \sqrt{2} \frac{t}{(F(t))^{\frac{3}{2}}} \frac{F(t)f'(t)}{(f(t))^2} \right] = 0,
\]

where we have used (10), (9) and the fact that \( F(t) \) has an exponential growth.
Let us consider now the terms of (41) containing $\alpha$. For $\delta$ small we have
\[
\frac{1}{\phi f(\phi)} \leq \frac{\delta}{\phi^2}. \tag{43}
\]
This estimate follows by (42).

We have (for $\delta$ small)
\[
\frac{\delta^3 f(\phi)}{\phi^3} \leq \frac{\delta}{\phi^2}. \tag{44}
\]
Indeed, rewrite (44) as
\[
\frac{\delta^2}{\phi(f(\phi))^{-1}} \leq 1.
\]
This estimate follows by the following result
\[
\lim_{\delta \to 0} \frac{\delta}{\phi^{\frac{3}{2}}(f(\phi))^{-\frac{1}{2}}} = \lim_{t \to \infty} \frac{\psi(t)}{t^{\frac{3}{2}}(f(t))^{-\frac{1}{2}}} = 0. \tag{45}
\]
In the last step of (45) we have used (10) and (9).

Therefore by (41), using (42), (43) and (44), we find suitable positive constants $M_1$, $M_2$ such that
\[
\Delta w < f(\phi) \left\{ 1 + H \delta + M_1 \frac{\delta}{\phi} + \alpha M_2 \frac{\delta}{\phi^2} \right\}. \tag{46}
\]
On the other side, using Taylor’s expansion we have
\[
f(w) = f(\phi) \left\{ 1 + \frac{f'(\phi)}{f(\phi)}(HC \delta + \alpha \delta \phi^{-1}) + \frac{f''(\overline{w})}{2f(\phi)}(HC \delta + \alpha \delta \phi^{-1})^2 \right\}, \tag{47}
\]
where
\[
\overline{w} = \phi + \theta(HC \delta + \alpha \delta \phi^{-1}), \quad 0 < \theta < 1.
\]
Let us take $\delta_0 > 0$ and $\alpha$ such that, for $\{ x \in \Omega : \delta(x) < \delta_0 \}$,
\[
-\epsilon < HC \delta + \alpha \delta \phi^{-1} < \epsilon, \tag{48}
\]
where $\epsilon$ is the constant appearing in condition (12). Then, using (11) (which follows from (9) and (10)) and (12), from (47) we get
\[
f(w) = f(\phi) \left\{ 1 + (1 + O(1)\frac{1}{\phi})(H \delta + \alpha \frac{\delta}{C \phi}) + O(1)(F(\phi))^{\frac{1}{2}}(H \delta + \alpha \frac{\delta}{C \phi}) \right\}.
\]
Denoting by $M_i$ suitable positive constants independent of $\alpha$ we find
\[
 f(w) > f(\phi) \left\{ 1 + H\delta + \alpha \frac{\delta}{C\phi} - M_3 \frac{\delta}{\phi} - \alpha M_4 \frac{\delta}{\phi^2} \right. \\
 \left. - M_5 (F(\phi))^{\frac{1}{m}} \delta^2 - M_6 (F(\phi))^{\frac{1}{m}} (\alpha \frac{\delta}{\phi})^2 \right\}. \tag{49}
\]

By (46) and (49) we find that
\[
 \Delta w < f(w) \tag{50}
\]
whenever
\[
 M_1 \frac{\delta}{\phi} + \alpha M_2 \frac{\delta}{\phi^2} < \alpha \frac{\delta}{C\phi} - M_3 \frac{\delta}{\phi} - \alpha M_4 \frac{\delta}{\phi^2} - M_5 (F(\phi))^{\frac{1}{m}} \delta^2 - M_6 (F(\phi))^{\frac{1}{m}} (\alpha \frac{\delta}{\phi})^2.
\]

Rearranging we find
\[
 M_1 + M_3 + M_5 (F(\phi))^{\frac{1}{m}} \delta \phi < \alpha \left[ \frac{1}{C} - (M_2 + M_4) \frac{1}{\phi} - M_6 (F(\phi))^{\frac{1}{m}} \alpha \frac{\delta}{\phi} \right]. \tag{51}
\]

We claim that
\[
 \lim_{\delta \to 0} (F(\phi))^{\frac{1}{m}} \delta \phi = 0. \tag{52}
\]
Indeed,
\[
 \lim_{\delta \to 0} (F(\phi))^{\frac{1}{m}} \delta \phi = \lim_{t \to \infty} \frac{\psi(t)}{t^{-1} (F(t))^{\frac{1}{m}}} \\
 = \lim_{t \to \infty} \frac{(2F(t))^{-\frac{1}{2}}}{t^{-2} (F(t))^{\frac{1}{m}} + t^{-1} \frac{1}{m} (F(t))^{-\frac{1}{m}} f(t)} = \frac{1}{\sqrt{2}} \lim_{t \to \infty} \frac{t (F(t))^{-\frac{1}{2}} + \frac{1}{m} f(t)}{t^{-1} + \frac{f(t)}{mF(t)}} = 0.
\]
The last step uses (1.10) and the limit
\[
 \lim_{t \to \infty} t (F(t))^{-\frac{1}{2}} + \frac{1}{m} f(t) = 0,
\]
true because $m > 2$ and $F(t)$ has an exponential growth. Estimate (52) is proved.

By using (10) one proves easily that $F(t)/t^2$ is increasing for large $t$. Moreover, if $G(t) = \int_0^t (F(\tau))^{\frac{1}{2}} d\tau$, using (9) we find
\[
 \lim_{t \to \infty} \frac{G(t)f(t)}{G'(t)F(t)} = \lim_{t \to \infty} \frac{G(t)}{(F(t))^{\frac{3}{2}} (f(t))^{-1}} = \lim_{t \to \infty} \frac{1}{3/2 - \frac{F(t)f(t)}{(F(t))^2}} = 2.
\]

Therefore (4) holds, and by (5) of Lemma 1.1 we have
\[
 M \delta^2 \phi'(\delta) + \phi(\delta) \leq u(x) \leq \phi(\delta) + M \delta \phi(\delta). \tag{53}
\]
Using de l’Hôpital rule and (1.10) we find

\[
\lim_{\delta \to 0} \frac{\delta \phi'(\delta)}{\phi(\delta)} = \lim_{t \to \infty} \frac{\psi(t)}{t(2F(t))^{1/2}}
\]

\[
= \lim_{t \to \infty} \frac{-(2F(t))^{1/2}}{(2F(t))^{1/2} - t(2F(t))^{1/2}f(t)} = \lim_{t \to \infty} \frac{2F(t) - f(t)}{f(t)} = 0.
\]

By this estimate, (53) implies

\[
-M \phi(\delta) \leq u(x) \leq \phi(\delta) + M \phi(\delta).
\]

Note that

\[
\lim_{\delta \to 0} \delta \phi(\delta) = \lim_{t \to \infty} \frac{\psi(t)}{t} = \lim_{t \to \infty} \frac{(2F(t))^{-1/2}}{t^{-2}} = 0.
\]

If \(M\) is the constant of (54), take \(\alpha\) and \(\delta_1\) so that

\[
\alpha \left( \frac{1}{\phi(\delta_1)} \right)^2 = 2M.
\]

In virtue of (52) we can increase \(\alpha\) and decrease \(\delta_1\) according to (55) so that (48) and (51) hold for \(\{x \in \Omega : \delta(x) < \delta_1\}\).

Using the right hand side of (54) we find

\[
w(x) - u(x) \geq \delta \phi \left[ H \frac{C}{\phi} + \alpha \frac{1}{\phi^2} - M \right].
\]

Decrease \(\delta_1\) if necessary (increasing \(\alpha\) according to (55)) until

\[
H \frac{C}{\phi} \geq -M
\]

for \(\delta(x) = \delta_1\). Then, (56) and (55) yield \(w(x) \geq u(x)\) on \(\{x \in \Omega : \delta(x) = \delta_1\}\).

On the other side, since \(\delta \phi(\delta) \to 0\) as \(\delta \to 0\), for \(\alpha\) fixed, by (56) we also get

\[
\liminf_{x \to \partial \Omega} [w(x) - u(x)] \geq 0.
\]

Hence, using (50), the equation \(\Delta u = f(u)\) and the comparison principle for elliptic equations ([8], Theorem 10.1) we find

\[
w(x) \geq u(x) \text{ in } \{x \in \Omega : \delta(x) < \delta_1\}.
\]

We look for a sub-solution of the form

\[
v(x) = \phi(\delta) + HC\delta - \alpha \delta(\phi(\delta))^{-1}.
\]
By the previous argument, instead of (46) now we find
\[ \Delta v > f(\phi) \left\{ 1 + H\delta - M_1 \frac{\delta}{\phi} - \alpha M_2 \frac{\delta}{\phi^2} \right\}, \quad (57) \]
and, instead of (49) we find
\[ f(v) < f(\phi) \left\{ 1 + H\delta - \alpha \frac{\delta}{C\phi} + M_3 \frac{\delta}{\phi} + \alpha M_4 \frac{\delta^2}{\phi^2} + M_5 (F(\phi)) \frac{1}{m} \delta + M_6 (F(\phi)) \frac{1}{m} (\alpha \delta \phi)^2 \right\}, \quad (58) \]
provided
\[ -\epsilon < HC\delta - \alpha \delta^{-1} < \epsilon. \quad (59) \]
Of course, the constants \( M_i \) in (57) and (58) are not necessarily the same as the constants of (46) and (49). By (57) and (58) we can find \( \alpha \) large and \( \delta_1 \) small so that (55) holds and so that \( \Delta v > f(v) \) in \( \{ x \in \Omega : \delta(x) < \delta_1 \} \). Using the left hand side of (54) we have
\[ v(x) - u(x) \leq \delta \phi \left[ \frac{HC}{\phi} - \alpha \frac{1}{\phi^2} + M \right]. \quad (60) \]
If we take \( \alpha \) and \( \delta_1 \) such that also
\[ HC \phi < M, \]
holds for \( \delta(x) = \delta_1 \), by (60) and (55) we find \( v(x) \leq u(x) \) on \( \{ x \in \Omega : \delta(x) = \delta_1 \} \) and
\[ \limsup_{x \to \partial \Omega} [v(x) - u(x)] \leq 0. \]
Using the comparison principle for elliptic equations we get
\[ v(x) \leq u(x) \quad \text{in} \quad \{ x \in \Omega : \delta(x) < \delta_1 \}. \]
Part a) of the theorem follows.

To prove part b), we look for a super-solution of the form
\[ w(x) = \phi(\delta) + H\delta g(\phi(\delta)) + \alpha \delta h(\phi(\delta)), \]
where \( H = (N - 1)K \), \( h(t) = g(t)|g'(t)| \) and \( \alpha \) is a positive constant to be determined. We have
\[ w_{x_i} = \phi' \delta_{x_i} + H_x \delta g(\phi) + H(g(\phi) + \delta g'(\phi)\phi') \delta_{x_i} + \alpha (h(\phi) + \delta h'(\phi)\phi') \delta_{x_i}, \]
and
\[ \Delta w = \phi'' - \phi' H + \Delta H \delta g(\phi) + 2\nabla H \cdot \nabla \delta (g(\phi) + \delta g'(\phi)\phi'). \]
Boundary blow-up solutions

\[ +H(2g'(\phi)\phi' + \delta g''(\phi)(\phi')^2 + \delta g'(\phi)\phi'') - H^2(g(\phi) + \delta g'(\phi)\phi') \]
\[ + \alpha(2h'(\phi)\phi' + \delta h''(\phi)(\phi')^2 + \delta h'(\phi)\phi'') - \alpha(h(\phi) + \delta h'(\phi)\phi')H. \]  

(61)

Equation (28) with \( s = \delta \) yields

\[ -\phi' = [1 + O(1)g'(\phi)]\delta f(\phi). \]  

(62)

Since \( \phi'' = f(\phi) \), by (61) and (62) we find

\[ \Delta w = f(\phi) \left\{ 1 + H\delta + O(1)\delta g'(\phi) + O(1)\frac{g'(\phi)}{f(\phi)} + O(1)\delta^3 g''(\phi) f(\phi) \right. \]
\[ + \alpha \left[ O(1)\delta g'(\phi)^2 + O(1)\delta g(\phi) g''(\phi) + O(1)\frac{g(\phi) g'(\phi)}{f(\phi)} \right] \]
\[ + O(1)\delta^3 g'(\phi) g''(\phi) f(\phi) + O(1)\delta^3 g(\phi) g'''(\phi) f(\phi) \].  

(63)

We claim that, for \( \delta \) small,

\[ \frac{g(\phi)}{f(\phi)} \leq \delta |g'(\phi)|. \]  

(64)

Suppose (15) holds, and rewrite (64) as

\[ \frac{g(\phi)}{\phi |g'(\phi)| \delta f(\phi)} \leq 1. \]

The latter inequality follows by (15) and the following result

\[ \lim_{\delta \to 0} \frac{\phi}{\delta f(\phi)} = \lim_{t \to \infty} \frac{t(f(t))^{-1}}{\psi(t)} = \lim_{t \to \infty} \frac{(f(t))^{-1} - t(f(t))^{-2} f'(t)}{-(2F(t))^{-\frac{1}{2}}} \]
\[ = \lim_{t \to \infty} \left[ -\frac{(2F(t))^{\frac{1}{2}}}{f(t)} + \frac{t}{(2F(t))^{\frac{1}{2}}} \frac{2F(t) f'(t)}{f(t)^2} \right] = 0. \]

In the last step we have used (34), (17) and the fact that, for a given \( p > 2 \) and \( t \) large, \( F(t) > t^p \). Estimate (64) follows in this case. If, instead, (16) holds then we write (64) as

\[ \frac{g(\phi)}{|g'(\phi)| \delta f(\phi)} \leq 1. \]

The latter inequality follows by (16) and the following result

\[ \lim_{\delta \to 0} \frac{1}{\delta f(\phi)} \leq \lim_{\delta \to 0} \frac{\phi}{\delta f(\phi)} = 0. \]
Estimate (64) is proved. 
Now we claim that, for δ small,
\[ \delta^3 |g''(\phi)| f(\phi) \leq \delta |g'(\phi)|. \] (65)
Suppose (15) holds, and rewrite (65) as
\[ \frac{\phi |g''(\phi)| \delta^2 f(\phi)}{|g'(\phi)|} \leq 1. \]
The latter inequality follows by (15) and the following result
\[
\lim_{\delta \to 0} \frac{\delta}{\phi^2(f(\phi))} = \lim_{t \to \infty} \frac{\psi(t)}{t^2(f(t))} = \lim_{t \to \infty} \frac{-2(2F(t))^{-\frac{1}{2}}}{(tf(t))^{-\frac{1}{2}} - t^2(f(t))^{-\frac{3}{2}} f'(t)}
\]
\[
= \lim_{t \to \infty} \frac{(\frac{2F(t)}{tf(t)})^{\frac{1}{2}}}{\frac{F(t)}{tf(t)} + \frac{F(t)(f'(t))}{(f(t))^2}} = 0,
\]
where (33) and (17) have been used. Now suppose (16) holds. The first inequality of (16) and (14) imply that \( g(t) \to 0 \) as \( t \to \infty \). Therefore (as already observed), (18) implies that
\[
\lim_{t \to \infty} \frac{F(t)}{f(t)} = 0.
\]
Let us write (65) as
\[ \frac{|g''(\phi)| \delta^2 f(\phi)}{|g'(\phi)|} \leq 1. \]
The latter inequality follows by (16) and the following result
\[
\lim_{\delta \to 0} \delta^{\frac{1}{2}} \left( \frac{f(\phi)}{f'(\phi)} \right)^{\frac{1}{2}} = \lim_{t \to \infty} \frac{\psi(t)}{(f(t))^{\frac{1}{2}}} = \lim_{t \to \infty} \frac{-2(2F(t))^{-\frac{1}{2}}}{(tf(t))^{-\frac{1}{2}} - t^2(f(t))^{-\frac{3}{2}} f'(t)}
\]
\[
= \lim_{t \to \infty} \frac{(\frac{2F(t)}{tf(t)})^{\frac{1}{2}}}{\frac{F(t)}{tf(t)} + \frac{F(t)(f'(t))}{(f(t))^2}} = 0,
\]
where (17) has been used again. Hence, (65) holds also in this case. By (64) and (65) it follows that the leading term which does not contain \( \alpha \) in (63) is \( O(1) \delta g'(\phi) \).
Let us consider now the terms containing \( \alpha \). For δ small we have
\[ \delta g(\phi)|g''(\phi)| \leq D^2 \delta |g'(\phi)|. \] (66)
This estimate follows by either (15) or (16).
We have (for $\delta$ small)
\[
\frac{g(\phi)|g'(\phi)|}{f(\phi)} \leq \delta(g'(\phi))^2.
\]
(67)
This follows by (64).

We have
\[
\delta^3 |g'(\phi)g''(\phi)| f(\phi) \leq \delta(g'(\phi))^2.
\]
(68)
This estimate follows by (65).

Finally, let us show that
\[
\delta^3 g(\phi) |g'''(\phi)| f(\phi) \leq \delta(g'(\phi))^2.
\]
(69)
If (15) holds, let us write (69) as
\[
\frac{g(\phi)}{\phi} \frac{\phi^2 |g'''(\phi)|}{|g'(\phi)|} \frac{\delta^2 f(\phi)}{\phi} \leq 1.
\]
The latter inequality follows by (15) and the following result
\[
\lim_{\delta \to 0} \frac{\delta}{\phi^{\frac{1}{2}} (f(\phi))^{-\frac{1}{2}}} = 0
\]
already proved. If (16) holds, let us write (69) as
\[
\frac{g(\phi)}{|g'''(\phi)|} \frac{|g''(\phi)|}{|g'(\phi)|} \delta^2 f(\phi) \leq 1.
\]
The latter inequality follows by (16) and the following result
\[
\lim_{\delta \to 0} \delta^2 f(\phi) = 0,
\]
already proved.

By (66)–(69) it follows that the leading term which contains $\alpha$ in (63) is $O(1)\delta(g'(\phi))^2$. Therefore, by (63), we find suitable positive constants $M_1$, $M_2$ such that
\[
\Delta w < f(\phi) \{1 + H\delta + M_1 \delta |g'(\phi)| + \alpha M_2 \delta(g'(\phi))^2\}.
\]
(70)
On the other side, using Taylor’s expansion we have
\[
f(w) = f(\phi) \left\{1 + \frac{f'(\phi)}{f(\phi)}(H\delta g(\phi) + \alpha \delta g(\phi)|g'(\phi)|) + \frac{f''(\phi)}{2f(\phi)}(H\delta g(\phi) + \alpha \delta g(\phi)|g'(\phi)|)^2\right\},
\]
(71)
where
\[ \overline{w} = \phi + \theta (H \delta g(\phi) + \alpha \delta g(\phi)|g'(\phi)|), \quad 0 < \theta < 1. \]
Let us take \( \delta_0 > 0 \) and \( \alpha \) such that, for \( \{ x \in \Omega : \delta(x) < \delta_0 \} \),
\[ -\epsilon < H \delta g(\phi) + \alpha \delta g(\phi)|g'(\phi)| < \epsilon, \quad (72) \]
where \( \epsilon \) is the constant appearing in condition (20). Then, using (19) and (20), from (71) we get
\[ f(w) = f(\phi) \left\{ 1 + (1 + O(1)g'(\phi))(H \delta + \alpha \delta |g'(\phi)|) \right\} + O(1)(F(\phi))^{\frac{1}{\alpha}}(H \delta + \alpha \delta |g'(\phi)|)^2. \]
Denoting by \( M_i \) suitable positive constants independent of \( \alpha \) we find
\[ f(w) > f(\phi)\{1 + H \delta + \alpha \delta |g'(\phi)| - M_3 \delta |g'(\phi)| - \alpha M_4 \delta (g'(\phi))^2 \]
\[ -M_5 (F(\phi))^{\frac{1}{\alpha} \delta^2} - M_6 (F(\phi))^{\frac{1}{\alpha}}(\alpha \delta g'(\phi))^2 \}. \quad (73) \]
By (70) and (73) we find that
\[ \Delta w < f(w) \quad (74) \]
when
\[ M_1 \delta |g'(\phi)| + \alpha M_2 \delta (g'(\phi))^2 < \alpha \delta |g'(\phi)| - M_3 \delta |g'(\phi)| \]
\[ -\alpha M_4 \delta (g'(\phi))^2 - M_5 (F(\phi))^{\frac{1}{\alpha} \delta^2} - M_6 (F(\phi))^{\frac{1}{\alpha}}(\alpha \delta g'(\phi))^2. \]
Rearranging we find
\[ M_1 + M_3 + M_5 (F(\phi))^{\frac{1}{\alpha} \delta} \frac{1}{|g'(\phi)|} \]
\[ < \alpha [1 - (M_2 + M_4)|g'(\phi)| - M_6 (F(\phi))^{\frac{1}{\alpha}} \alpha \delta |g'(\phi)|]. \quad (75) \]
We claim that
\[ \lim_{\delta \to 0} (F(\phi))^{\frac{1}{\alpha} \delta} \frac{1}{g'(\phi)} = 0. \quad (76) \]
Indeed, if (15) holds we have
\[ 0 \leq \lim_{\delta \to 0} (F(\phi))^{\frac{1}{\alpha} \delta} \frac{1}{|g'(\phi)|} \leq D \lim_{\delta \to 0} (F(\phi))^{\frac{1}{\alpha} \delta} \frac{\phi}{g(\phi)} \]
\[ = D \lim_{t \to \infty} \frac{\psi(t)}{t^{-1}g(t)(F(t))^{-\frac{1}{m}}} = \frac{D}{\sqrt{2}} \lim_{t \to \infty} \frac{-t(F(t))^{-\frac{1}{2} + \frac{1}{m}}}{-t^{-1}g(t) + g'(t) - \frac{1}{m}g(t) \frac{f(t)}{F(t)}} = 0. \]
The last step uses (14), (15), (18) and the limit
\[
\lim_{t \to \infty} t(F(t))^{-\frac{1}{2} + \frac{1}{m}} = 0,
\]
true because \( m > 2 \) and \( F(t) > t^p \) for any \( p > 1 \) and \( t \) large. Estimate (76) follows in this case. If (16) holds then we have
\[
\lim_{\delta \to 0} (F(\phi))^{\frac{1}{m}} \delta \frac{1}{g'(\phi)} \leq D \lim_{\delta \to 0} (F(\phi))^{\frac{1}{m}} \frac{1}{g(\phi)}
\]
\[
= D \lim_{t \to \infty} \frac{\psi(t)}{g(t)(F(t))^{-\frac{1}{2} + \frac{1}{m}}} = \frac{D}{\sqrt{2}} \lim_{t \to \infty} \frac{-\frac{1}{2} t}{g'(t) - \frac{1}{m} g(t) f(t)} = 0.
\]
Estimate (76) is proved.

The previous proof shows also that
\[
\lim_{\delta \to 0} (F(\phi))^{\frac{1}{m}} \delta \frac{\phi}{g(\phi)} = 0.
\]
(77)

As in the proof of part a), if \( G(t) = \int_0^t (F(\tau))^\frac{1}{2} d\tau \), we find
\[
\lim_{t \to \infty} G(t) f(t) = 2.
\]
Therefore (4) holds and by (5) of Lemma 1.1 we have
\[
M \delta^2 \phi'(\delta) + \phi(\delta) \leq u(x) \leq \phi(\delta) + M \delta \phi(\delta).
\]
(78)

Using de l'Hôpital rule and (33) we find
\[
\lim_{\delta \to 0} \frac{\delta |\phi'(\delta)|}{\phi(\delta)} = \lim_{t \to \infty} \frac{\psi(t)}{t(2F(t))^{-\frac{1}{2}}}
\]
\[
= \lim_{t \to \infty} \frac{-\frac{1}{2} (2F(t))^{\frac{1}{2}}}{(2F(t))^{-\frac{1}{2}} - t(2F(t))^{-\frac{1}{2}} f(t)} = \lim_{t \to \infty} \frac{-\frac{2F(t)}{2F(t) - 1}}{tf(t)} = 0.
\]
Therefore, (78) implies
\[
-M \delta \phi(\delta) + \phi(\delta) \leq u(x) \leq \phi(\delta) + M \delta \phi(\delta).
\]
(79)

If \( M \) is the constant of (79), take \( \alpha \) and \( \delta_1 \) so that
\[
\alpha g'(\phi(\delta_1)) \frac{g(\phi(\delta_1))}{\phi(\delta_1)} = 2M.
\]
(80)
In virtue of (76) and (77) we can increase $\alpha$ and decrease $\delta_1$ according to (80) so that (72) and (75) hold for $\{x \in \Omega : \delta(x) < \delta_1\}$.

Using the right hand side of (79) we find

$$w(x) - u(x) \geq \delta \phi \left[ \frac{Hg(\phi)}{\phi} + \alpha |g'(\phi)| \frac{g(\phi)}{\phi} - M \right].$$

(81)

Decrease $\delta_1$ if necessary (increasing $\alpha$ according to (80)) until

$$H \frac{g(\phi)}{\phi} \geq -M$$

for $\delta(x) = \delta_1$. Then, (81) and (80) yield $w(x) \geq u(x)$ on $\{x \in \Omega : \delta(x) = \delta_1\}$.

Since $\delta \phi(\delta) \to 0$ as $\delta \to 0$, for $\alpha$ fixed, by (81) we also get

$$\liminf_{x \to \partial \Omega} [w(x) - u(x)] \geq 0.$$ 

Hence, using (74), the equation $\Delta u = f(u)$ and the comparison principle we find

$$w(x) \geq u(x) \text{ in } \{x \in \Omega : \delta(x) < \delta_1\}.$$ 

To finish, we look for a sub-solution of the form

$$v(x) = \phi(\delta) + H\delta g(\phi) - \alpha \delta g(\phi)|g'(\phi)|,$$

where $\alpha$ is a positive constant to be determined. Instead of (70), now we find

$$\Delta v > f(\phi) \left\{ 1 + H\delta - M_1 \delta |g'(\phi)| - \alpha M_2 \delta (g'(\phi))^2 \right\}.$$ 

(82)

Of course, the constants $M_i$ in (82) and in what follows are not necessarily the same as in the previous case. Using Taylor’s expansion, instead of (73) we find

$$f(v) < f(\phi) \left\{ 1 + H\delta - \alpha \delta |g'(\phi)| + M_3 \delta |g'(\phi)| + \alpha M_4 \delta (g'(\phi))^2 \right\}$$

$$+ M_5 (F(\phi)) \frac{1}{\pi} \delta^2 + M_6 (F(\phi)) \frac{1}{\pi} (\alpha \delta g'(\phi))^2 \right\},$$ 

(83)

provided

$$-\epsilon < H\delta g(\phi) - \alpha \delta g(\phi)|g'(\phi)| < \epsilon,$$ 

(84)

where $\epsilon$ is the constant appearing in condition (20). By (82) and (83) we find that

$$\Delta v > f(v)$$ 

(85)

when

$$-M_1 \delta |g'(\phi)| - \alpha M_2 \delta (g'(\phi))^2 > -\alpha \delta |g'(\phi)| + M_3 \delta |g'(\phi)|$$

$$+ \alpha M_4 \delta (g'(\phi))^2 + M_5 (F(\phi)) \frac{1}{\pi} \delta^2 + M_6 (F(\phi)) \frac{1}{\pi} (\alpha \delta g'(\phi))^2.$$
Rearranging we find

\[ M_1 + M_3 + M_5(F(\phi)) \frac{1}{|g'(\phi)|} \delta \frac{1}{g'(\phi)} \]

\[ < \alpha[1 - (M_2 + M_4)|g'(\phi)| - M_6(F(\phi))]^\frac{1}{m} \alpha \delta |g'(\phi)|. \]  

(86)

This inequality is the same as (75) (possibly with different value of the constants \( M_1 \)). Arguing exactly as in the previous case we find \( \alpha \) and \( \delta_1 \) such that (84) and (86) hold in \( \{x \in \Omega : \delta(x) < \delta_1\} \).

Using the left hand side of (79) we find

\[ v(x) - u(x) \leq \delta \phi \left[ H \frac{g(\phi)}{\phi} - \alpha |g'(\phi)| \frac{g(\phi)}{\phi} + M \right]. \]  

(87)

Decrease \( \delta_1 \) increasing \( \alpha \) according to (80) so that

\[ H \frac{g(\phi)}{\phi} \leq M \]

for \( \delta(x) = \delta_1 \). Then, (87) and (80) yield \( v(x) \leq u(x) \) on \( \{x \in \Omega : \delta(x) = \delta_1\} \).

Since \( \delta \phi(\delta) \to 0 \) as \( \delta \to 0 \), by (87) (when \( \alpha \) is fixed) we also get

\[ \limsup_{x \to \partial \Omega} [v(x) - u(x)] \leq 0. \]

Hence, using (85), the equation \( \Delta u = f(u) \) and the comparison principle we find

\[ v(x) \leq u(x) \quad \text{in} \quad \{x \in \Omega : \delta(x) < \delta_1\}. \]

Part b) of the theorem follows.

References


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