A New Class of Bivariate Gompertz Distributions and its Mixture\textsuperscript{1}

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Abstract

A new class of bivariate Gompertz distributions is presented in this paper. The model introduced here is of Marshall-Olkin type. The used procedure is based on a latent random variable with exponential distribution. A mixture of the suggested bivariate distributions is also derived. The obtained results in this paper generalize those of Marshall-Olkin bivariate exponential distribution and other present in the literature.

Keywords: Bivariate Gompertz distribution, Moment generating function, Mixture distributions

1 Introduction

The Gompertz distribution plays an important role in modelling human mortality and fitting actuarial tables. This distribution was first introduced by Gompertz [1]. It has been used as a growth model and also used to fit the tumor growth. The Gompertz distribution is related by a simple transformation to certain distribution in the family of distributions obtained by Pearson. Applications and more recent survey of the Gompertz distribution can be found in [2].

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In many practical situations, multivariate lifetime data arise frequently, and in these situations it is important to consider different multivariate models that could be used to model such multivariate lifetime data. The model introduced in this paper is of some interest, in reliability theory, for example, sometimes failure rate can occur for more than one reason and a mixture distribution is nice tool for modelling such situation.

In fact, shock models are used in reliability to describe different applications. Shocks can refer for example to damage caused to biological organs by illness or environmental causes of damage acting on a technical system, see for example El-Gohary [4, 5, 6], and A-hameed and Proschan [7]. Also El-Gohary and Al-Ruzaiza have obtained a new class of bivariate distribution with pareto of Marshall-Olkin type [3].

The objective of this paper is to introduce a new class of bivariate Gompertz distributions of Marsall-Olkin type. It is considered as a distribution of the life times of two dependent components each has a Gompertz distribution. Also the mixture of the proposed Gompertz distributions will be derived.

The paper is organized as follows. Section 2 presents the shock model yielding the bivariate Gompertz distribution. The joint survival and probability density function of bivariate proposed Gompertz distribution is derived. Section 3 presents the joint moment generating function of this bivariate distribution and its marginal moment generating functions. Section 4 discusses the mixture of proposed bivariate Gompertz distributions and its moment generating function.

2  The new class of bivariate Gompertz distributions

In this section, we define a new class of bivariate Gompertz distribution using shock models. We start with the joint survival function of the proposed bivariate distribution and so used it to derive the corresponding joint probability density function. The marginal probability density functions and conditional probability density functions of this distribution are also derived. Finally this contains mathematical expectation of this distribution.
2.1 The joint survival function

Assume that there exists a three independent sources of shocks are presented in the environment of a system consists of two components [7]. A shock from source 1 destroys the component 1; it occurs at a random time $T_1$. A shock from source 2 destroys component 2; it occurs at a random time $T_2$. A shock from source 3 destroys both the components; it occurs at a random time $T_3$. Thus the random lifetime of the component 1, say $X_1$, satisfies $X_1 = \min(T_1, T_3)$. While the random lifetime of component 2, say $X_2$, satisfies $X_2 = \min(T_2, T_3)$ [9].

Let us assume that the random variables $T_1$ and $T_2$ having Gompertz distribution with parameters $\alpha_1, \beta_1$ and $\alpha_2, \beta_2$ respectively while the random variable $T_3$ has an exponential distribution with parameter $\theta$, shortly we say $\text{Gomp}(\alpha_i, \beta_i)$, $i = 1, 2$ and $T_3$ has $\text{Exp}(\theta)$ [3].

That is, the probability density function of the random lifetime $T_i$, $i = 1, 2$, takes the following form

$$g_i(t) = \alpha_i e^{\beta_i t} \bar{G}_i(t), \quad t \geq 0, \quad \alpha_i > 0, \quad \beta_i > 0, \quad (i = 1, 2) \quad (2.1)$$

where $\bar{G}_i(t)$ is the survival function of $T_i$, $i = 1, 2$, which is given by

$$\bar{G}_i(t) = \exp \left\{ - k_i \left( e^{\beta_i t} - 1 \right) \right\}, \quad \alpha_i = k_i \beta_i, \quad (i = 1, 2) \quad (2.2)$$

The probability density function of $T_3$ takes the following form

$$g_3(t) = \theta \bar{G}_3(t), \quad t \geq 0, \quad \theta > 0. \quad (2.3)$$

where $\bar{G}_3(t)$ is survival function of $T_3$ which given by

$$\bar{G}_3(t) = e^{-\theta t}. \quad (2.4)$$

Obviously, the random variables $X_1$ and $X_2$ are dependent because of the common source of shock 3.

Now we proceed to investigate the joint survival function of the random variables $X_1$ and $X_2$. The following lemma presents the joint survival function $F_{X_1,X_2}(x_1, x_2)$ of these variables.

**Lemma 2.1** The joint survival function of $X_1$ and $X_2$ is

$$\bar{F}_{X_1,X_2}(x_1, x_2) = \bar{G}_3(z) \prod_{i=1}^2 \bar{G}_i(x_i) = \exp \left[ -k_1 \left( e^{\beta_1 x_1} - 1 \right) - k_2 \left( e^{\beta_2 x_2} - 1 \right) - \theta z \right] \quad (2.5)$$
where $z = \max(x_1, x_2)$.

**Proof.** Since the joint survival function of $X_1$ and $X_2$ is defined as

$$F_{X_1, X_2}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$$

Then

$$F_{X_1, X_2}(x_1, x_2) = P\left(\left\{ \min(T_1, T_3) > x_1 \right\}, \left\{ \min(T_2, T_3) > x_2 \right\}\right)$$

$$= P\left(\left\{ T_1 > x_1, T_3 > x_2 \right\}, \left\{ T_2 > x_1, T_3 > x_2 \right\}\right)$$

$$= P\left( T_1 > x_1, T_2 > x_2, T_3 > \max(x_1, x_2) \right)$$

As the random variables $T_j, (j = 1, 2, 3)$ are mutually independent, we directly obtain

$$F_{X_1, X_2}(x_1, x_2) = P(T_1 > x_1) P(T_2 > x_2) P(T_3 > \max(x_1, x_2))$$

$$= \bar{G}_1(x_1) \bar{G}_2(x_2) \bar{G}_3(z), \quad z = \max(x_1, x_2)$$

Substituting from (2.2) and (2.4) into the above relation, we can reach the form (2.5) that completes the proof.

The following Corollary gives the marginal survival functions of the random variable $X_1$ and $X_2$.

**Corollary 2.1** The marginal survival functions of bivariate Gompertz distribution are given by

$$F_{X_i}(x_i) = G_i(x_i) \bar{G}_3(x_i) = \exp \left[ -k_i (e^{\beta_i x_i} - 1) - \theta x_i \right], \quad i = 1, 2$$

(2.6)

**Proof.** The proof of this Corollary can be done in a similar manner as in the proof of Lemma 2.1.

The following Corollary presents the joint distribution of $X_1$ and $X_2$.

**Corollary 2.2** The joint distribution function $F_{X_1, X_2}(x_1, x_2)$ of $X_1$ and $X_2$ is given by

$$F_{X_1, X_2}(x_1, x_2) = 1 - \bar{G}_1(x_1) \bar{G}_3(x_1) - \bar{G}_2(x_2) \bar{G}_3(x_2) + \bar{G}_3(z) \prod_{i=1}^{2} \bar{G}_i(x_i)$$

$$= 1 - \exp \left[ -k_1 (e^{\beta_1 x_1} - 1) - \theta x_1 \right] - \exp \left[ -k_2 (e^{\beta_2 x_2} - 1) - \theta x_2 \right] + \exp \left[ -k_1 (e^{\beta_1 x_1} - 1) - k_2 (e^{\beta_2 x_2} - 1) - \theta z \right]$$

(2.7)

**Proof.** The proof of this Corollary can be reached by using (2.5) and (2.6) with the help of the following relation:

$$F_{X_1, X_2}(x_1, x_2) = 1 - \bar{F}_{X_1}(x_1) - \bar{F}_{X_2}(x_2) + \bar{F}_{X_1, X_2}(x_1, x_2).$$

We can easily observe that the joint survival function of Marshall-Olkin bivariate exponential distribution [8, 9, 10] can be obtained by setting $\beta_i, (i = 1, 2)$ tend to zero.
2.2 The joint probability density function

The following theorem provides the joint probability density function of the new bivariate Gompertz distribution.

**Theorem 2.1** If the joint survival function of \(X_1\) and \(X_2\) is given by

\[
\bar{F}_{X_1, X_2}(x_1, x_2) = \exp \left[ -k_1 \left( e^{\beta_1 x_1} - 1 \right) - k_2 \left( e^{\beta_2 x_2} - 1 \right) - \theta z \right]
\]  

(2.8)

Then, then joint probability density function \(f_{X_1, X_2}(x_1, x_2)\) of \(X_1\) and \(X_2\) takes the form

\[
f_{X_1, X_2}(x_1, x_2) = \begin{cases} 
    f_1(x_1, x_2) & \text{if } x_1 > x_2 > 0, \\
    f_2(x_1, x_2) & \text{if } x_2 > x_1 > 0, \\
    f_0(x, x) & \text{if } x_1 = x_2 = x > 0
\end{cases}
\]  

(2.9)

where

\[
f_1(x_1, x_2) = k_2 \beta_2 e^{\beta_2 x_2} (\theta + k_1 \beta_1 e^{\beta_1 x_1}) \exp \left[ -k_1 \left( e^{\beta_1 x_1} - 1 \right) - k_2 \left( e^{\beta_2 x_2} - 1 \right) - \theta x_1 \right]
\]

\[
f_2(x_1, x_2) = k_1 \beta_1 e^{\beta_1 x_1} (\theta + k_2 \beta_2 e^{\beta_2 x_2}) \exp \left[ -k_1 \left( e^{\beta_1 x_1} - 1 \right) - k_2 \left( e^{\beta_2 x_2} - 1 \right) - \theta x_2 \right]
\]

\[
f_0(x, x) = \theta \exp \left[ -k_1 \left( e^{\beta_1 x} - 1 \right) - k_2 \left( e^{\beta_2 x} - 1 \right) - \theta x \right]
\]  

(2.10)

**Proof.** The proof of this Theorem is based on the obtaining forms of \(f_1(x_1, x_2)\) and \(f_2(x_1, x_2)\) by differentiating the joint survival function \(\bar{F}_{X_1, X_2}(x, y)\) with respect to \(x_1\) and \(x_2\), that is

\[
f_{X_1, X_2}(x_1, x_2) = \begin{cases} 
    \frac{\partial^2 \bar{F}(x_1, x_2)}{\partial x_1 \partial x_2} & \text{if } x_1 > x_2 > 0, \\
    \frac{\partial^2 \bar{F}(x_1, x_2)}{\partial x_1 \partial x_2} & \text{if } x_2 > x_1 > 0,
\end{cases}
\]

But \(f_0(x, x)\) can not be derived in a similar method. Instead we use the following identity to derive \(f_0(x, x)\).

\[
\int_0^\infty \int_0^{x_1} f_1(x_1, x_2)dx_1dx_2 + \int_0^\infty \int_0^{x_2} f_2(x_1, x_2)dx_1dx_2 + \int_0^\infty f_0(x, x)dx = 1
\]  

(2.11)

One can find out that

\[
\int_0^\infty \int_0^{x_1} f_1(x_1, x_2)dx_2dx_1 = 1 - \int_0^\infty \left( \theta + k_1 \beta_1 e^{\beta_1 x_1} \right) \bar{F}_{X_1}(x_1)dx_1
\]  

(2.12)

and

\[
\int_0^\infty \int_0^{x_2} f_2(x_1, x_2)dx_1dx_2 = 1 - \int_0^\infty \left( \theta + k_2 \beta_2 e^{\beta_2 x_2} \right) \bar{F}_{X_2}(x_2)dx_2
\]  

(2.13)
Substituting from (2.10) and (2.11) into (2.9) we obtain

\[ \int_0^\infty f_0(x, x) dx = \theta \int_0^\infty \prod_{i=1}^3 G_i(x) dx \]

Thus, \( f_0(x, x) \) is

\[ f_0(x, x) = \theta \exp \left[ -k_1 \left( e^{\beta_1 x} - 1 \right) - k_2 \beta_2 \left( e^{\beta_2 x} - 1 \right) - \theta x \right], \quad x > 0. \]

which completes the proof.

This class of bivariate distributions generalizes the bivariate distribution of Marshall-Olkin [8]. This result shows that the results of this paper generalize those of Marshall-Olkin bivariate exponential distribution. Further the survival function of bivariate Weibull distributions can be derived using the nonlinear transformation \( X_i = \ln(1 + Y_i^{\beta_i}) / \beta_i, \quad (i = 1, 2) \) [12]. That is

\[ \tilde{F}(y_1, y_2) = e^{-(k_1 y_1^{\beta_1} + k_1 y_2^{\beta_2} + \theta z)}, \quad z = \max(y_1, y_2). \]

Therefore this survival function can be used to derive the probability density function of bivariate Weibull distribution in [12].

Next, plots of the joint density function (2.9) for some selected values of distribution parameters are shown in figures 1 – 4.

![Figures 1a and 1b](image-url)

Figures 1a and 1b. A plot of the joint density function \( f(x_1, x_2) \), Equation (2.9) for \( \alpha_2 = 2\alpha_1 = 0.4, \beta_1 = \beta_2 = 10 \) and \( \theta = 5 \).
Figures 2a and 2b. A plot of the joint density function $f(x_1, x_2)$, Equation (2.9) for $\alpha_1 = \alpha_2 = 0.4, \beta_1 = 40\beta_2 = 200$ and $\theta = 5$.

Figures 3a and 3b. A plot of the joint density function $f(x_1, x_2)$, Equation (2.9) for $\alpha_1 = \alpha_2 = 2, \beta_2 = 100\beta_1 = 200$ and $\theta = 5$. 
Figures 4a and 4b. A plot of the joint density function \( f(x_1, x_2) \), Equation (2.9) for \( \alpha_1 = \alpha_2 = 2, \beta_2 = 1500, \beta_1 = 2 \) and \( \theta = 25 \).

A comparison of Figures 3b and 4b shows the relative rate at which the density tail of.

Lemma 2.2 The joint probability density function of the Marshall-Olkin bivariate exponential distribution is [8, 11]

\[
f_{X_1,X_2}(x_1, x_2) = \begin{cases} 
\alpha_2(\alpha_1 + \theta) \exp \left[ - (\alpha_1 + \theta)x_1 - \alpha_2 x_2 \right], & x_1 > x_2 > 0 \\
\alpha_1(\alpha_2 + \theta) \exp \left[ - k_i \beta_i x_1 - (\alpha_2 + \theta) x_2 \right], & x_2 > x_1 > 0 
\end{cases}
\]

Also, this distribution has a mass of \( \theta/(\theta + \alpha_1 + \alpha_2) \) along the diagonal \( x_1 = x_2 \).

Proof. The result of this lemma can be obtained immediately from theorem (2.1) upon setting \( \beta_1 \) and \( \beta_2 \) tend to zero.

\( \square \)

2.3 Marginal probability density functions

The following Corollary gives the marginal probability density functions of \( X_1 \) and \( X_2 \).

Theorem 2.2 The marginal pdf of \( X_i \), \( (i = 1, 2) \) is given by

\[
f_{X_i}(x_i) = (\theta + k_i \beta_i e^{\beta_i x_i}) \exp \left[ -k_i \left( e^{\beta_i x_i} - 1 \right) - \theta x_i \right], \ x_i > 0, \ (i = 1, 2) \quad (2.14)
\]
Proof. First we derive \( f_X(x) \) using the fact that
\[
f_X(x) = \int_0^\infty f_{X_1X_2}(x_1, x_2) \, dx_2 = \int_0^\infty f_1(x_1, x_2) \, dx_2 + \int_{x_1}^\infty f_2(x_1, x_2) \, dx_2 + f_0(x_1, x_1)
\]
Using the expressions of \( f_1(x_1, x_2) \), \( f_2(x_1, x_2) \) and \( f_0(x_1, x_1) \) given in Theorem (2.1), we can get \( f_X(x) \) of the form (2.14). Proceeding similarly, we can derive \( f_X_2(x_2) \) as given in (2.14), which completes the proof of the theorem. \( \square \)

Lemma 2.3 The marginal probability density function of \( X_i \), \( i = 1, 2 \) of Marshall-Olkin bivariate exponential distribution is
\[
f_X(x) = (\theta + \alpha) e^{-(\theta+\alpha)x}, \quad x > 0, \quad (i = 1, 2)
\]
Proof. The proof of this lemma can be obtained immediately from theorem (2.2) upon setting \( \beta_i, (i = 1, 2) \) tend to zero. \( \square \)

2.4 Conditional probability density functions

Theorem 2.3 The conditional probability density function of \( X_i \) given \( X_j = x_j \), \( i, j = 1, 2, i \neq j \) is given by
\[
f_{X_{ixj}}(x_i|x_j) = \begin{cases} 
  f_{X_{ixj}}^{(1)}(x_i|x_j) & \text{if } x_i > x_j > 0, \\
  f_{X_{ixj}}^{(2)}(x_i|x_j) & \text{if } x_j > x_i > 0, \\
  f_{X_{ixj}}^{(0)}(x_i|x_j) & \text{if } x_i = x_j > 0
\end{cases}
\]
where
\[
f_{X_{ixj}}^{(1)}(x_i|x_j) = k_j \beta_j \left( \theta + k_i \beta_i e^{\beta_i x_i} \right) \exp \left\{ -k_i \left( e^{\beta_i x_i} - 1 \right) \theta x_i \left( \theta + \beta_j x_j \right) \right\} / \left( \theta + k_j \beta_j e^{\beta_j x_j} \right)
\]
\[
f_{X_{ixj}}^{(2)}(x_i|x_j) = k_i \beta_i \exp \left\{ -k_i \left( e^{\beta_i x_i} - 1 \right) \beta_i x_i \right\},
\]
\[
f_{X_{ixj}}^{(0)}(x_i|x_j) = \theta \exp \left\{ -k_i \left( e^{\beta_i x_i} - 1 \right) \right\} / \left( \theta + k_j \beta_j e^{\beta_j x_j} \right).
\]
Proof. The theorem follows readily upon substituting for the joint probability density function of \( (X_1, X_2) \) in (2.10) and the marginal probability density function of \( X_i \), \( (i = 1, 2) \) in (2.14), the following relation
\[
f_{X_{ixj}}(x_i|x_j) = \frac{f_{X_{ixj}}(x_i, x_j)}{f_X(x_i)}, \quad (i = 1, 2)
\]
(2.16)
which completes the proof. \qed

**Lemma 2.1** For the case of Marshall-Olkin bivariate exponential distribution, we can obtain by setting $\beta_1$ and $\beta_2$ tend to zero in (2.15).

$$f_{X_i|X_j}(x_i|x_j) = \begin{cases} 
\alpha_j e^{-(\theta + \alpha_i)x_i + \theta x_j} & \text{if } x_i > x_j > 0, \\
\alpha_i e^{-\alpha_i x_i} & \text{if } x_j > x_i > 0, \\
\theta e^{-\alpha_i x_i}/(\theta + \alpha_j) & \text{if } x_i = x_j > 0
\end{cases} \quad (2.17)$$

**Proof.** The proof of this lemma can be reach readily by setting $\beta_i, (i = 1, 2)$ tend to zero in (2.15) which completes the proof. \qed

### 2.5 Mathematical expectation

This subsection presents the exact forms of mathematical expectation of $X_i, (i = 1, 2)$, and the second moments of $X_i^2, (i = 1, 2)$ and $X_1X_2$.

**Theorem 2.4** The expectation of $X_i, (i = 1, 2)$ is given

$$E(X_i) = k_i^{\theta/\beta_i} e^{k_i \Gamma(-\theta/\beta_i, k_i)/\beta_i}, \quad (i = 1, 2) \quad (2.18)$$

where $\Gamma(\mu, \nu)$ is the incomplete gamma function that defined by

$$\Gamma(\mu, \nu) = \int_{\nu}^{\infty} t^{\mu-1} e^{-t} dt, \quad t > 0$$

which for fixed $\nu$ is an entire function of $\mu$. For non-integer $\mu$ this function is a multi-valued function of $\nu$ with a branch point at $\nu = 0$.

**Proof.** Starting with

$$E(X_i) = \int_{0}^{\infty} x_i f_{X_i}(x_i) dx_i$$

and substituting $f_{X_i}(x_i)$ from (2.14), we get

$$E(X_i) = \int_{0}^{\infty} x_i (\theta + k_i \beta_i e^{\beta x_i}) \exp \left[-k_i (e^{\beta x_i} - 1) - \theta x_i \right] dx_i$$

$$= \int_{0}^{\infty} \exp \left[-k_i (e^{\beta x_i} - 1) - \theta x_i \right] dx_i = e^{k_i} \int_{0}^{\infty} e^{-\theta x_i} \exp \left[-k_i e^{\beta x_i} \right] dx_i$$

and using the incomplete gamma function definition (see Gradshteyn and Ryzhik 3.331 p.308 and 8.350 p.940 [11]) we derive the expression in (2.18), which completes the proof. \qed
Theorem 2.5 The expectation of $X_i^2$ for $i = 1, 2$ is given

$$E(X_i^2) = 2k_i^{\theta/\beta_i} e^{k_i} \left[ \partial \Gamma(1-\theta/\beta_i, k_i) / \partial \theta - \ln k_i \Gamma(1-\theta/\beta_i, k_i) \right] / \beta_i^2, \quad (i = 1, 2)$$

(2.19)

Proof. Starting with

$$E(X_i^2) = \int_0^\infty x_i^2 f_{X_i}(x_i) \, dx_i$$

and substituting $f_{X_i}(x_i)$ from (2.14), and putting $u = k_i e^{\beta_i x_i}$, we get

$$E(X_i^2) = 2e^{k_i} k_i^{\theta/\beta_i} \left[ \int_{k_i}^{\infty} \frac{\ln u e^{-u}}{u^{(\theta/\beta_i+1)}} \, du \right] \Gamma(0, k_i) \Gamma(1-\theta/\beta_i, k_i)$$

and using the incomplete gamma function definition and its differentiation with respect to one of their parameters (see Gradshteyn and Ryzhik 4.358 p.578 and 8.350 p.940 [11]) we derive the expression in (2.19), which completes the proof.


Theorem 2.6 The expectation of $X_1 X_2$ is given

$$E(X_1 X_2) = e^{k_1+k_2} \left\{ k_1^{\theta/\beta_1} \Gamma(0, k_2) \Gamma(1-\theta/\beta_1, k_1) - (\Psi(1) + \ln k_2) \Gamma(1-\theta/\beta_1, k_1) + 2\beta_2 \ln k_1 \Gamma(1-\theta/\beta_1, k_1) / \beta_1 + 2\beta_2 \partial \Gamma(1-\theta/\beta_1, k_1) / \partial \theta - \beta_1 \beta_2 \Gamma(1-\theta/\beta_1, k_1) / \theta 

- \beta_2 \ln k_1 \Gamma(1-\theta/\beta_1, k_1) \right\} + k_2^{\theta/\beta_2} \Gamma(0, k_1) \Gamma(1-\theta/\beta_2) + \sum_{n=1}^{\infty} (-1)^n k_1^{(\theta-n\beta_2)/\beta_1} k_2^n \times

\left[ n\beta_1 \partial \Gamma((n\beta_2 - \theta)/\beta_1 + 1, k_1) / \partial \theta - (n\beta_2 - \theta) \beta_2 \partial \Gamma((n\beta_2 - \theta)/\beta_1 + 1, k_1) / \partial \theta + n \ln k_1 \Gamma((n\beta_2 - \theta)/\beta_1 + 1, k_1) \beta_1 - (n\beta_2 - \theta)(\beta_1 - n\beta_2 \ln k_1) \Gamma((n\beta_2 - \theta)/\beta_1 + 1, k_1) \right] / \beta_1 \beta_2 \right\} / \beta_1 \beta_2$$

(2.20)

where $\Psi(x)$ is the digamma function.

Proof. Starting with

$$E(X_1 X_2) = \int_0^\infty \int_0^{x_1} x_1 x_2 f_1(x_1, x_2) \, dx_2 \, dx_1 + \int_0^\infty \int_0^\infty x_1 x_2 f_2(x_1, x_2) \, dx_2 \, dx_1 + \int_0^\infty x^2 f_0(x, x) \, dx$$

and substituting $f_1(x_1, x_2), f_2(x_1, x_2)$ and $f_0(x_1, x_2)$ from (2.10) and using the integral properties we get
\[ E(X_1X_2) = \int_0^\infty \int_0^{x_1} x_1(x_1 + k_1 \beta_1 e^{x_1}) \exp \left\{ -k_1 e^{x_1} - k_2 e^{x_2} - \theta x_1 \right\} dx_2 dx_1 + k_1 \beta_1 \int_0^\infty \int_0^{x_1} x_1 e^{x_1} \exp \left\{ -k_1 e^{x_1} - k_2 e^{x_2} - \theta x_2 \right\} dx_2 dx_1 \]

using the definition of the incomplete gamma function and its series form (see Gradshteyn and Ryzhik 8.212 p.925, 8.214 p.927 and 8.354 p.941 [11]) and after a lengthly algebraic manipulation we derive the expression in (2.20). which completes the proof. \(\square\)

**Lemma 2.2** For the Marshall-Olkin bivariate exponential, we have

\[ E(X_i) = \frac{1}{\theta + \alpha_i}, \quad E(X_i^2) = \frac{2}{(\theta + \alpha_i)^2}, \quad (i = 1, 2) \quad (2.21) \]

and

\[ E(X_1X_2) = \frac{1}{\alpha_2(\alpha_1 + \theta)} + \frac{\alpha_1}{(\theta + \alpha_2)(\theta + \alpha_1 + \alpha_2)^2} - \frac{\theta + \alpha_1}{\alpha_2(\theta + \alpha_1 + \alpha_2)^2} \quad (2.22) \]

By taking limits of the expressions (2.18), (2.19) and (2.20) as \(\beta_1\) and \(\beta_2\) tend to zero we can get the expressions (2.21) and (2.22).

### 3 Moment generating functions

In this section we present the joint moment generating function of \((X_1, X_2)\) and marginal moment generating function of \(X_i, (i = 1, 2)\).

**Theorem 3.1** The joint moment generating function of \((X_1, X_2)\) is given by:

\[
M_{X_1,X_2}(t_1, t_2) = 1 - e^{k_1 + k_2} \left\{ t_1 k_1^{(t_1 + \theta)/\beta_1} \Gamma(-t_1/\beta_1 - \theta/\beta_1, k_1)/\beta_1 + t_2 k_2^{(t_2 + \theta)/\beta_2} \Gamma(-t_2/\beta_2, k_2) - \Gamma(-t_2/\beta_2) \right\} \\
\left[ e^{-k_1} - t_1 k_1^{(\theta + t_1)/\beta_1} \Gamma(-t_1 + \theta/\beta_1, k_1)/\beta_1 \right] + t_2 k_2^{(\theta + t_2)/\beta_2} \Gamma(-t_2 + \theta/\beta_2, k_2) \\
\sum_{n=0}^{\infty} (-1)^n k_2^n \left[ e^{-k_1} \left( (n\beta_2 - t_2)(n\beta_2 - t_2 - \theta) \right)^{n-1} - k_1^{(t_1 + t_2 + \theta - n\beta_2)/\beta_1} \Gamma(t_1 + t_2 - \theta - n\beta_2, k_1) \right]
\]

\[
(t_1 + t_2 - n\beta_2) \Gamma(-t_1 + t_2 + \theta - n\beta_2/\beta_1, k_1) \left( \beta_1(n\beta_2 - \theta - t_2) \right)^{-1} + k_1^{(t_1 + t_2 - n\beta_2)/\beta_1} \times \\
(t_1 + t_2 - n\beta_2) \Gamma(-t_1 + t_2 + \theta - n\beta_2/\beta_1, k_1) \left( \beta_1(n\beta_2 - \theta - t_2) \right)^{-1} / (n!n) \}
\]

(3.1)
Proof. Starting with, the definition moment generating function of bivariate distribution of \((X_1, X_2)\) we have
\[
M_{X_1, X_2}(t_1, t_2) = E \left( e^{-(t_1 X_1 + t_2 X_2)} \right)
\]
That is,
\[
M_{X_1, X_2}(t_1, t_2) = \int_0^\infty \int_0^{x_2} e^{-(t_1 x_1 + t_2 x_2)} f_1(x_1, x_2) \, dx_2 \, dx_1 + \int_0^\infty \int_0^{x_1} e^{-(t_1 x_1 + t_2 x_2)} f_2(x_1, x_2) \, dx_1 \, dx_2 + \int_0^\infty e^{-(t_1 + t_2)x} f_0(x, x) \, dx
\]
Substituting from (2.9) into the above relation and after a lengthy algebraic manipulation we can reach the form (3.1).
The following lemma presents the marginal moment generating function of \(X_i, (i = 1, 2)\).

**Lemma 3.1** The marginal moment generating function of \(X_i, (i = 1, 2)\) is given by:
\[
M_i(t_i) = \left[ 1 - t_i e^{k_i (t_i + \theta)/\beta_i} \Gamma \left(- (t_i + \theta)/\beta_i, k_i/\beta_i\right) \right], \quad (i = 1, 2) \quad (3.2)
\]

Proof. The proof of this lemma can be done by using either: the relation between the joint and marginal moment generating functions, or the definition of the marginal moment generating function of the random variable. Note that the moment generating function of the bivariate Gompertz can be used to derive the mathematical expectations \(X_i, X_2^2, \quad i = 1, 2\) and \(X_1 X_2\).

Next, we will discuss a form of the bivariate mixture of Gompertz distribution of Marshall-Olkin type where the dependence among the components is characterized by a latent variable independently distributed of the individual components.

### 4 Mixture of Bivariate Gompertz Distributions

In this section, we present the mixture of independent Gompertz distributions. Then we derive a mixture of bivariate Gompertz distributions where the dependence among the components is characterized by a latent random variable which is independently exponentially distributed of the individual component.
Consider a system of two components where the lifetime of component \( i, i = 1, 2 \), say, \( X_i \), is a mixture of two independent Gompertz distributions. That is,

\[
X_1 \sim a_1 \text{Gomp}(\alpha_{11}, \beta_{11}) + (1 - a_1) \text{Gomp}(\alpha_{12}, \beta_{12}), \quad 0 \leq a_1 \leq 1,
\]

and

\[
X_2 \sim a_2 \text{Gomp}(\alpha_{21}, \beta_{21}) + (1 - a_2) \text{Gomp}(\alpha_{22}, \beta_{22}), \quad 0 \leq a_2 \leq 1.
\]

The notation \( \text{Gomp}(\alpha_{ij}, \beta_{ij}) \) means a random variable, say \( X_{ij} \), having a Gompertz distribution with the parameters \( \alpha_{ij}, \beta_{ij} \). That is the probability density function of \( X_{ij} \) takes the following form

\[
f_{X_{ij}}(x) = \alpha_{ij} e^{\beta_{ij} x} \exp \left\{ -k_{ij}(e^{\beta_{ij} x} - 1) \right\}, \quad x > 0, \quad \alpha_{ij} = k_{ij} \beta_{ij} > 0, \quad \beta_{ij} > 0, \quad \forall i, j.
\]

Consider also an exponentially distributed random variable, say \( Z \), with parameter \( \theta \) which is independent of \( X_{ij} \) for all \( i, j \). The random variable \( Z \) will be used as a latent variable to introduce dependence among \( X \)'s. The density function of \( Z \) is

\[
f_Z(z) = \theta e^{-\theta z}, \quad z > 0, \quad \theta > 0.
\]

Using the independence assumption in the above model, we can see that \( Z \) is also independent of \( X_1 \) and \( X_2 \).

Define \( S_i = \min(X_i, Z) \) for \( i = 1, 2 \). Then the vector \( S = (S_1, S_2) \) follows a bivariate distribution and obviously they are dependent as they commonly share the influence of the latent random variable \( Z \).

In what follows we present the joint probability density function of \( (S_1, S_2) \). Firstly, we derive the joint survival function of \( (S_1, S_2) \). Then we use it to derive the joint probability density function of the mixture of bivariate Gompertz distribution.

**Corollary 4.1** The joint survival function of \( S_1, S_2 \) is

\[
\bar{F}_{S_1,S_2}(s_1, s_2) = p_{11} \exp \left[ -k_{11}(e^{\beta_{11}s_1} - 1) - k_{21}(e^{\beta_{21}s_2} - 1) - \theta s \right]
\]

\[
+ p_{12} \exp \left[ -k_{11}(e^{\beta_{11}s_1} - 1) - k_{22}(e^{\beta_{22}s_2} - 1) - \theta s \right]
\]

\[
+ p_{12} \exp \left[ -k_{12}(e^{\beta_{12}s_1} - 1) - k_{21}(e^{\beta_{21}s_2} - 1) - \theta s \right]
\]

\[
+ p_{22} \left[ -k_{12}(e^{\beta_{12}s_1} - 1) - k_{22}(e^{\beta_{22}s_2} - 1) - \theta s \right]
\]

(4.1)

where \( s = \max(s_1, s_2) > 0 \) and for \( i, j \in \{1, 2\} \):

\[
p_{ij} = a_i^{2-i} a_j^{2-j} (1 - a_1)^{i-1} (1 - a_2)^{j-1}.
\]
Theorem 4.1

The joint pdf of $S_1, S_2$ is given by (3.1), that completes the proof.

The following Theorem gives the joint probability density function of a mixture of four bivariate Gompertz distributions.

**Proof.** Since

$$F_{S_1,S_2}(s_1, s_2) = P(S_1 > s_1, S_2 > s_2)$$

Then using the definitions of $S_1$ and $S_2$ we have

$$F_{S_1,S_2}(s_1, s_2) = P(X_1 > s_1) P(X_2 > s_2) P(Z > s_0) =$$

$$e^{-\theta_2} \prod_{i=1}^{2} \{a_i \exp \left[ -k_{i1}(e^{\beta_{i1}s_i} - 1) \right]$$

$$+ (1-a_i) \exp \left[ -k_{i2}(e^{\beta_{i2}s_i} - 1) \right] \}$$

One can write the above relation as given by (3.1), that completes the proof. 

Note that:

1. For $i, j \in \{1, 2\}, p_{ij} \geq 0$ and $p_{11} + p_{12} + p_{21} + p_{22} = 1$.

2. Each term function in the right hand side of (4.1) is a survival function of the new bivariate Gompertz distribution.

This means that, survival function given in (4.1) is a joint survival function of a mixture of four bivariate Gompertz distributions.

The following Theorem gives the joint probability density function of $(S_1, S_2)$.

**Theorem 4.1** The joint pdf of $S_1, S_2$, say $f(s_1, s_2)$, is

$$f(s_1, s_2) = \begin{cases} 
  f_1(s_1, s_2) & \text{if } s_1 > s_2 > 0, \\
  f_2(s_1, s_2) & \text{if } s_2 > s_1 > 0, \\
  f_0(s,s) & \text{if } s_1 = s_2 = s > 0,
\end{cases}$$

where

$$f_1(s_1, s_2) = p_{11}k_{21}e^{\beta_{21}s_2} (\theta + k_{11}e^{\beta_{11}s_1}) \exp \left\{ -k_{11}(e^{\beta_{11}s_1} - 1) - k_{21}(e^{\beta_{21}s_2} - 1) - \theta s_1 \right\}$$

$$+ p_{12}k_{22}e^{\beta_{22}s_2} (\theta + k_{11}e^{\beta_{11}s_1}) \exp \left\{ -k_{11}(e^{\beta_{11}s_1} - 1) - k_{22}(e^{\beta_{22}s_2} - 1) - \theta s_1 \right\}$$

$$+ p_{21}k_{21}e^{\beta_{21}s_2} (\theta + k_{12}e^{\beta_{12}s_1}) \exp \left\{ -k_{12}(e^{\beta_{12}s_1} - 1) - k_{21}(e^{\beta_{21}s_2} - 1) - \theta s_1 \right\}$$

$$+ p_{22}k_{22}e^{\beta_{22}s_2} (\theta + k_{12}e^{\beta_{12}s_1}) \exp \left\{ -k_{12}(e^{\beta_{12}s_1} - 1) - k_{22}(e^{\beta_{22}s_2} - 1) - \theta s_1 \right\}$$

$$f_2(s_1, s_2) = p_{11}k_{11}e^{\beta_{11}s_1} (\theta + k_{21}e^{\beta_{21}s_2}) \exp \left\{ -k_{11}(e^{\beta_{11}s_1} - 1) - k_{21}(e^{\beta_{21}s_2} - 1) - \theta s_2 \right\}$$

$$+ p_{12}k_{11}e^{\beta_{11}s_1} (\theta + k_{22}e^{\beta_{22}s_2}) \exp \left\{ -k_{11}(e^{\beta_{11}s_1} - 1) - k_{22}(e^{\beta_{22}s_2} - 1) - \theta s_2 \right\}$$

$$+ p_{21}k_{12}e^{\beta_{12}s_1} (\theta + k_{21}e^{\beta_{21}s_2}) \exp \left\{ -k_{12}(e^{\beta_{12}s_1} - 1) - k_{21}(e^{\beta_{21}s_2} - 1) - \theta s_2 \right\}$$

$$+ p_{22}k_{12}e^{\beta_{12}s_1} (\theta + k_{22}e^{\beta_{22}s_2}) \exp \left\{ -k_{12}(e^{\beta_{12}s_1} - 1) - k_{22}(e^{\beta_{22}s_2} - 1) - \theta s_2 \right\}$$
The following Corollary gives the marginal pdf’s of $S_2$ (2.1).

**Lemma 4.3**

From the marginal densities, we can derive the marginal moment generating functions of $S_1$ and $S_2$.

**Theorem**

The proof of this Theorem follows along the same lines as of theorem (2.1).

**Proof.**

The following Corollary gives the marginal pdf’s of $S_1$ and $S_2$.

**Lemma 4.1**

The marginal density functions of $S_1$ and $S_2$ are respectively,

\[ f_{S_1}(s_1) = a_1(\theta + k_{11}\beta_{11}e^{\theta s_1}) \exp \left\{ -k_{11}(\beta_{11}s_1 - 1) - \theta s_1 \right\} \]

\[ + (1 - a_1)(\theta + k_{12}\beta_{12}e^{\theta s_1}) \exp \left\{ -k_{12}(\beta_{12}s_1 - 1) - \theta s_2 \right\}, \quad s_1 > 0, \]

and

\[ f_{S_2}(s_2) = a_2(\theta + k_{21}\beta_{21}e^{\theta s_2}) \exp \left\{ -k_{21}(\beta_{21}s_2 - 1) - \theta s_2 \right\} \]

\[ + (1 - a_2)(\theta + k_{22}\beta_{22}e^{\theta s_2}) \exp \left\{ -k_{22}(\beta_{22}s_2 - 1) - \theta s_2 \right\}, \quad s_2 > 0. \]

From the marginal densities, we can derive the marginal moment generating functions of $S_i$, ($i = 1, 2$) as follows.

**Lemma 4.2**

The moment generating functions of $X_1$ and $X_2$ are respectively,

\[ M_{S_1}(t_1) = 1 - t_1 \left\{ a_1 e^{k_{11}k_{11}(\theta + t_1)/\beta_{11}}(1 - (\theta + t_1)/\beta_{11}, k_{11})/\beta_{11} + \right\} \]

\[ (1 - a_1) e^{k_{12}k_{12}(\theta + t_1)/\beta_{12}}(1 - (\theta + t_1)/\beta_{12}, k_{12})/\beta_{12} \] \hspace{1cm} (4.9)

and

\[ M_{S_2}(t_2) = 1 - t_2 \left\{ a_2 e^{k_{21}k_{21}(\theta + t_2)/\beta_{21}}(1 - (\theta + t_2)/\beta_{21}, k_{21})/\beta_{21} + \right\} \]

\[ (1 - a_2) e^{k_{22}k_{22}(\theta + t_2)/\beta_{22}}(1 - (\theta + t_2)/\beta_{22}, k_{22})/\beta_{22} \] \hspace{1cm} (4.10)

**Lemma 4.3**

From (4.9) and (4.10), we readily have

\[ E(S_1) = a_1 e^{k_{11}k_{11}(\theta + t_1)/\beta_{11}}(1 - a_1) e^{k_{12}k_{12}(\theta + t_1)/\beta_{12}} \]

\[ \Gamma(\theta + t_1, k_{11})/\beta_{11} + \Gamma(\theta + t_2, k_{21})/\beta_{21} \] \hspace{1cm} (4.11)
and
\[
E(S_2) = a_2 e^{k_21} k_2^{\theta/\beta_21} \Gamma\left(-\theta/\beta_21, k_21\right)/\beta_21 + (1-a_2) e^{k_22} k_2^{\theta/\beta_22} \Gamma\left(-\theta/\beta_22, k_22\right)/\beta_22
\]

Lemma 4.4 From (4.9) and (4.10), we readily have
\[
E(S_1^2) = -2 a_1 e^{k_{11}} \left[ \beta_{11} \partial \Gamma\left(-\theta/\beta_{11}, k_{11}\right)/\partial \theta + \ln k_{11} \Gamma\left(-\theta/\beta_{11}, k_{11}\right) \right] / \beta_{11}^2
\]
\[
-2(1-a_1) e^{k_{12}} \left[ \beta_{12} \partial \Gamma\left(-\theta/\beta_{12}, k_{12}\right)/\partial \theta + \ln k_{12} \Gamma\left(-\theta/\beta_{12}, k_{12}\right) \right] / \beta_{12}^2
\]
and
\[
E(S_2^2) = -2 a_2 e^{k_{21}} \left[ \beta_{21} \partial \Gamma\left(-\theta/\beta_{21}, k_{21}\right)/\partial \theta + \ln k_{21} \Gamma\left(-\theta/\beta_{21}, k_{21}\right) \right] / \beta_{21}^2
\]
\[
-2(1-a_2) e^{k_{22}} \left[ \beta_{22} \partial \Gamma\left(-\theta/\beta_{22}, k_{22}\right)/\partial \theta + \ln k_{22} \Gamma\left(-\theta/\beta_{22}, k_{22}\right) \right] / \beta_{22}^2
\]

Lemma 4.5 The joint moment generating function of \((S_1, S_2)\) is given by
\[
M_{S_1, S_2}(t_1, t_2) = \sum_i^4 b_i M_i(t_1, t_2)
\]
where
\[
\begin{align*}
 b_1 &= p_{11}, \quad b_2 = p_{12}, \quad b_3 = p_{21}, \quad b_4 = p_{22} \\
 M_1 &= M_{X_{11}, X_{21}}(t_1, t_2), \quad M_2 = M_{X_{11}, X_{22}}(t_1, t_2), \quad M_3 = M_{X_{12}, X_{21}}(t_1, t_2), \quad M_4 = M_{X_{12}, X_{22}}(t_1, t_2)
\end{align*}
\]
and the functions \(M_{X_{11}, X_{21}}(t_1, t_2), M_{X_{11}, X_{22}}(t_1, t_2), M_{X_{12}, X_{21}}(t_1, t_2)\) and
\(M_{X_{12}, X_{22}}(t_1, t_2)\) can be derived from (2.14) respectively, by replacing \(\alpha_1, \alpha_2, \beta_1, \beta_2\)
with \(\alpha_{11}, \alpha_{21}, \beta_{11}, \beta_{21}; \alpha_{11}, \alpha_{22}, \beta_{11}, \beta_{22};\alpha_{12}, \alpha_{21}, \beta_{12}, \beta_{21}\) and \(\alpha_{12}, \alpha_{22}, \beta_{12}, \beta_{22}.\)

Proof. One can establish this lemma from (3.1) and (4.15).

5 Conclusion

Finally we conclude that a new class of bivariate Gompertz distribution of Marshall-Olkin type is derived. The used procedure is based on a latent random variable with exponential distribution. The moment generating functions of bivariate new class is derived. A mixture of the suggested bivariate distributions is derived. The present results generalize those of Marshall-Olkin bivariate exponential distribution and other present in the literature [5, 8, 10].
References


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